Abelian groups

Group structure

You are reminded that a group is an abstract mathematical structure comprising a set G and a binary operation o that satisfies the following properties (axioms).

(1) Identity

There is an element $e \in G$ such that for any element $a \in G$

$$e \circ a = a$$
$$a \circ e = a$$

(2) Inverses

For every element $a \in G$ there is another element $a^{-1} \in G$ such that

$$a \circ a^{-1} = e$$
$$a^{-1} \circ a = e$$

(3) Closure

For every pair of elements $a,b, \in G$ there is a third element $a \circ b \in G$

(4) Associativity

For all elements $a,b,c \in G$

$$(a \circ b) \circ c = a \circ (b \circ c)$$

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The meaning of commutative

The above properties of identity, inverses, closure and associativity are the minimum requirements to define an algebraic structure. A group is the starting point for the study of algebraic structure for this reason. However, there can exist algebraic structures with more structure, not less. One thing to consider adding to a group structure is the property of commutativity.

Commutativity is the property that the order in which we do things does not matter. Some very familiar operations, such as addition and multiplication, have this property, and consequently, we take it for granted.

For example the order in which we add two numbers does not matter – whatever order we do this we will get to the same total. Thus $3+4=7$ and $4+3=7$; so we can swap $3+4$ with $4+3$ interchangeably. The same applies for multiplication:

$4\times 3=12=3\times 4$

Addition and multiplication are examples of binary operations and

$G=\{\mathbb{R},+\}$ and $H=\{\mathbb{R},\times\}$

are groups. Let a and b be any two arbitrary real numbers. Then we have indicated that

$$a+b=b+a$$
$$a\times b=b\times a$$

hold. Thus, both G and H are groups which the binary operation in question is commutative. They are commutative groups – we also call them Abelian groups.

In general, a binary operation $\circ$ is commutative if, for any two elements a and b

$$a\circ b=b\circ a$$

The property of commutativity is not trivial – by this, we mean, that there are structures that fail to be commutative. The usual standard example is matrix multiplication. For matrices it is *not* true that the order in which you multiply them together does not matter. In general, for matrices, $AB\neq BA$. We will show this with an example:

Example

Let $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $B=\begin{pmatrix}2&3\\4&5\end{pmatrix}$

Then $AB=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}2&3\\4&5\end{pmatrix}=\begin{pmatrix}10&13\\22&29\end{pmatrix}$

But $BA=\begin{pmatrix}2&3\\4&5\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}11&16\\19&28\end{pmatrix}$

Hence $AB\neq BA$

Hence, adding the property of commutativity to a group turns it into a new structure meriting a separate name of Abelian group (which means, a commutative group).

Example

Prove $\left(\mathbb{R}_{+} \setminus \{1\}, *\right)$ is an Abelian group, where $x * y = x^{\ln y}$.

Solution

Firstly, we remind you that the expression $\mathbb{R}_{+} \setminus \{1\}$ means the set of positive real numbers with the number 1 removed from it. To prove that this is a group we have to verify all the axioms – that is the four group axioms of identity, inverses, closure and associativity, together with the additional axiom of commutativity.

(a) Identity

For the identity element, i, we require

$x * i = x$ for all $x \in \mathbb{R}_{+}^{*} \setminus \{1\}$

This implies $x^{\ln i} = x$

Hence

$\ln i = 1$

$i = e$

Therefore, the number e is the identity element

(b) Inverses

To find the inverse, x^{-1} of x we follow through a series of equivalences

$$
\begin{aligned}
x * x^{-1} = e \quad &\Leftrightarrow \quad x^{\ln x^{-1}} = e \\
&\Leftrightarrow \ln\left(x^{\ln x^{-1}}\right) = \ln e \\
&\Leftrightarrow \quad \ln x^{-1} \times \ln x = 1 \\
&\Leftrightarrow \quad \ln x^{-1} = \frac{1}{\ln x} \\
&\Rightarrow \quad x^{-1} = e^{\frac{1}{\ln x}}.
\end{aligned}
$$

Now if $x \neq 1$ then $x^{-1} \in \mathbb{R}_{+} \setminus \{1\}$

$x^{-1} = e^{\frac{1}{\ln x}}$ is the inverse of the $x \in \mathbb{R}_{+} \setminus \{1\}$

(c) Closure

For all $x, y \in \mathbb{R}_{+} \setminus \{1\}$

$x * y = x^{\ln y} \neq 1$ and $x^{\ln y} \in \mathbb{R}_{+} \setminus \{1\}$

(d) Associativity

For all $x, y, z \in \mathbb{R}_{+}^{*} \setminus \{1\}$

$$\left(x * y\right) * z = \left(x^{\ln y} * z\right) = \left(x^{\ln y}\right)^{\ln z} = x^{\ln y \ln z} = x^{\ln y^{\ln z}} = x^{\ln(y * z)} = x * \left(y * z\right)$$

(e) <u>Commutativity</u>

Note that $x^y = e^{y \ln x}$

Hence $x^{\ln y} = e^{\ln x . \ln y}$

Likewise $y^{\ln x} = e^{\ln x . \ln y}$

Thus, for all $x, y \in \mathbb{R}_+^* \setminus \{1\}$

$x * y = x^{\ln y} = e^{\ln y \ln x} = y^{\ln x} = y * x$

Hence $\left(\mathbb{R}_+ \setminus \{1\}, *\right)$ is an Abelian group.