Applications of Differentiation

Problem solving with the differential calculus

The differential calculus has many practical applications in engineering. Students may have wondered in the past why, when they visit a supermarket, the shapes of all the tin cans are virtually identical. This is an application of the differential calculus, which determines the optimum shape of the can to use the least amount of metal in its manufacture.

This is illustrated by the following example.

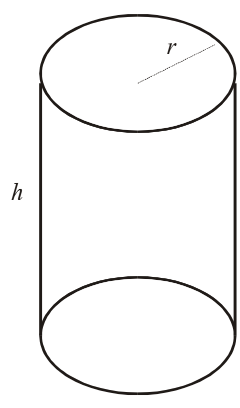
Example

(a) A company requires closed cylindrical tins of height h cm and radius r cm. The sides of the tin can be made from rectangular sheets of width h cm and length $2\pi r$ cm without waste. Stamping out each circular end of area πr^2 cm^2 requires an area of $4r^2$ cm^2 and some is wasted. If the tins are to contain 8000cm^3:

(i) Find h in terms of r to give the correct volume.
(ii) Find the area of tinplate required for each tin in terms of r.

(b) If tinplate costs 0.03p per cm^2, find (by calculus methods) the radius, height and cost of the tin that will contain the required volume most cheaply.

Solution



(a) (i) $V = 8000 \text{ cm}^3$

The volume is given by the usual formula; here the base is πr^2 and the height is h

$$V = \pi r^2 h = 8000$$

Rearrangement gives

$$h = \frac{8000}{\pi r^2}$$

(ii) The area is the sum of the rectangle making up the side, with area $2\pi rh$ and the two circular end-pieces, with area $(2r)^2 = 4r^2$ each. Hence

$$A = 2\pi rh + 8r^2$$

On substituting $h = \frac{8000}{\pi r^2}$ into this equation we obtain

$$A = 2\pi r \times \frac{8000}{\pi r^2} + 8r^2$$

$$A = \frac{16000}{r} + 8r^2$$

To find the minimum area required we need to find the turning points of this function. This requires us to differentiate the function with respect to r.

We can write the area as a function of r

$$A = f(r) = \frac{16000}{r} + 8r^2$$

Putting this in index notation

$$A = f(r) = 16000r^{-1} + 8r^2$$

And differentiating

$$\frac{dA}{dr} = f'(r) = -16000r^{-2} + 16r$$

For turning points $\frac{dA}{dr} = f'(r) = 0$

Thus

$$16r - \frac{16000}{r^2} = 0$$

$$16r^3 = 16000$$
$$r^3 = 1000$$
$$r = 10 \text{ cm}$$

Strictly speaking we should show that this is a minimum area. Actually, glancing at the function

$$A = \frac{16000}{r} + 8r^2$$

shows that as r gets very big the area increases because of the r^2 term, and as r tends to zero, the area also increases because of the $\frac{1}{r}$ term. Thus, the turning point is a minimum.

(b) The cost at this minimum value is given by

$$\begin{aligned} \text{cost} &= 0.03\left(\frac{16000}{10} + 8 \times 10^2\right) \\ &= 0.03 \times 2400 \\ &= 72p \end{aligned}$$

Analytical curve sketching and the determination of roots

A great number of practical problems in physics and business result in the formation of an equation of the form

$$f(x) = 0$$

It is then required to solve this equation. It may not always be possible to use an algebraic technique to find its solution. There is a formula for all quadratics, and quartic (beginning with x^4) polynomials of the form

$ax^4 + ... +$ terms in descending powers of x +... + constant = 0

can be reduced to quadratics. There is also a formula for the solution to any cubic equation. However, there is no general formula for the quintic, and functions involving the "transcendental" functions, that is the trigonometric, exponential and logarithmic, functions do not generally have formulae. Thus, in this case a mixture of analytic and numerical methods must be used. The calculus is used to *sketch* the function, and then a numerical method is used to find the root. This process is illustrated by the following example

Example

Show that the equation $x^5 + 4x^3 + x + 11 = 0$ has a single root.

Solution

Firstly, we use the differential calculus to sketch the curve. We must extract information about its turning points and behaviour as x approaches infinity.

At turning points $\frac{df}{dx} = 0$, where

$$f(x) = x^5 + 4x^3 + x + 11$$

Hence

$$f'(x) = 5x^4 + 12x^2 + 1$$

And for turning points

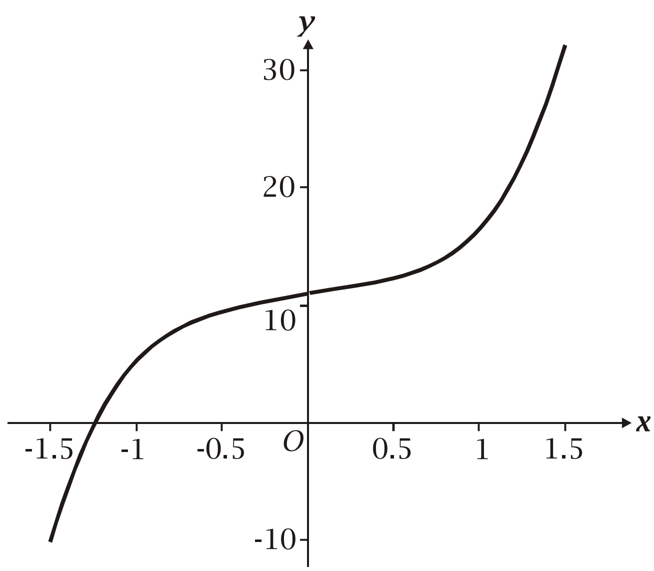
$$5x^4 + 12x^2 + 1 = 0$$

The roots of this equation are obtained in the following way. We denote by $y = x^2$, and we get a second order equation $5y + 12y + 1 = 0$.

$$y_{1,2} = \frac{-12 \pm \sqrt{144-20}}{10} = \begin{cases} \dfrac{-12+\sqrt{124}}{10} \\ \dfrac{-12-\sqrt{124}}{10} \end{cases}$$

But we observe that both y_1 and y_2 are negative, and $y_{1,2} = x^2$ which is a contradiction. Therefore, there is no turning point.

As $x \to -\infty,\ y \to -\infty$, and as $x \to \infty,\ y \to \infty$, so the graph is always increasing.
At $x = 0,\ y = 11$
Therefore, the curve $y = x^5 + 4x^3 + x + 11$ look like this



Therefore, we have a single root, which lies in the interval $(-2,-1)$, since $y(-2) < 0$ and $y(-1) > 0$. To find that root we now use a numerical method, such as the method of trial and error.

$y(x) = x^5 + 4x^3 + x + 11$

$y(-2) = -55$

$y(-1) = 5$ $\qquad -2 < \alpha < -1$

$y(-1.5) = 11.593...$ $\qquad -1.5 < \alpha < -1$

$y(-1.2) = 0.399...$ $\qquad -1.5 < \alpha < -1.2$

$y(-1.3) = -2.800...$ $\qquad -1.3 < \alpha < -1.2$

$y(-1.25) - 1.114...$ $\qquad -1.25 < \alpha < 1.20$

Therefore, the root is

$\alpha = 1.2(1.D.P.)$