Applications of integration to find areas

Areas

You should already be familiar with the idea of using integration to find the area under a curve.

Example (1)

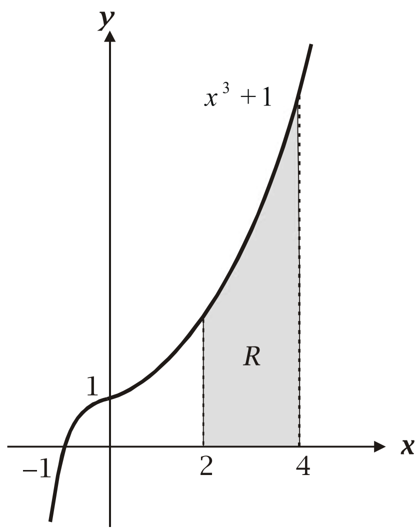
An area, *R*, is enclosed between the curve $y = x^3 + 1$, the lines $x = 2$ to $x = 4$ and the *x*-axis.

(*a*) Make a sketch the curve $y = x^3 + 1$ for $-3 \le x \le 6$ and mark the region *R* on it.

(*b*) Find the area of *R* to 3 significant figures.

Solution

(*a*)



(*b*) The area *R* is the integral of $y = x^3 + 1$ from $x = 2$ to $x = 4$.

$$R = \int_2^4 x^3 + 1 \, dx$$

$$= \left[\frac{x^4}{4} + x \right]_2^4$$

$$= \left(\frac{4^4}{4} + 4 \right) - \left(\frac{2^4}{4} + 2 \right) = 68 - 6 = 62 \text{ sq units}$$

The next problem will illustrate a new aspect to this process of finding areas.

Example (2)

(*a*) Evaluate the integral

$$I = \int_{-2}^{-1} x^3 + 1 \, dx$$

(*b*) By sketching the region bounded by the curve $y = x^3 + 1$ the lines $x = -2,\ x = -1$ and the *x*-axis interpret the meaning of the negative sign obtained in part (*a*).

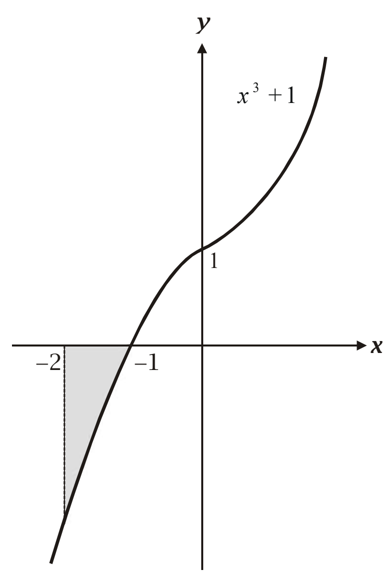
Solution

(*a*)

$$\begin{aligned} I &= \int_{-2}^{-1} x^3 + 1 \, dx \\ &= \left[\frac{x^4}{4} + x \right]_{-2}^{-1} \\ &= \left(\frac{(-1)^4}{4} - 1 \right) - \left(\frac{(-2)^4}{4} - 2 \right) \\ &= -\frac{3}{4} - (4 - 2) \\ &= -2\frac{3}{4} \end{aligned}$$

This is a negative value.

(*b*) The following is a sketch of the region bounded by $y = x^3 + 1$ and the lines $x = -2,\ x = -1$ and the *x*-axis.



This shows that the region lies below the *x*-axis. This explains the negative sign in the integral $I = \int_{-2}^{-1} x^3 + 1 \, dx$.

This shows that the integral provides a *signed value for the area*, meaning that an area below the x-axis takes a negative sign, and an area above the x-axis takes a positive sign. Areas are naturally only positive quantities. Thus, if you are asked to find the area of a region lying below the x-axis, you integrate as normal, but at the end of the problem you remove the negative sign. The area found in this last example, show above in the diagram, is $2\frac{3}{4}$ square units, and not $-2\frac{3}{4}$ square units, because negative areas only have meaning in the context of integration, where a positive sign means "area above the x-axis" and a negative sign means "area below the x-axis". The fact that integrals can take negative values, whereas areas cannot, means that you have to take care in some questions. This is because integral can cancel out.

Example (3)

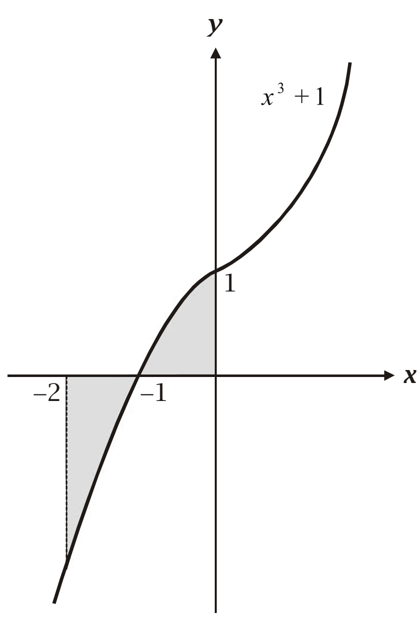
Find the area, R, enclosed between the curve $y = x^3 + 1$, the lines $x = -2$, $x = 0$ and the x-axis.

Solution

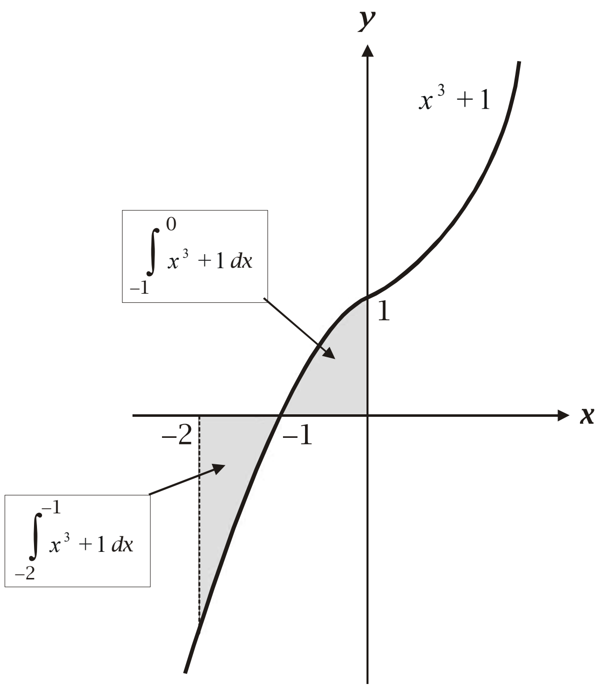
The natural thing to do here is to find the integral of $y = x^3 + 1$ from $x = -2$ to $x = 0$

$$I = \int_{-2}^{0} x^3 + 1\, dx = \left[\frac{x^4}{4} + x\right]_{-2}^{0} = 0 - \left(\frac{(-2)^4}{4} - 2\right) = -2 \ (2 \text{ sq. units})$$

We found the integral of $I = \int_{-2}^{-1} x^3 + 1\, dx$ to be $-2\frac{3}{4}\left(2\frac{3}{4} \text{ sq. units}\right)$ so although the region of integration is larger the area appears to have shrunk! This cannot be right. A sketch of $y = x^3 + 1$ with the region R shows us where we are going wrong.



The region *R* is the whole of the shaded area in the diagram above, and comprises a section above the *x*-axis and below the *x*-axis. The integral supplies a signed area, and the area of the portion below the *x*-axis is subtracted from the area above the *x*-axis. Consequently, the part of one area cancels out the other. In this case the net signed area is negative, indicating that more of the area lies below the *x*-axis than above it.



As the diagram shows, the thing to do is to cut the region up into *two* pieces, one lying below the x –axis and one lying above it. Then integrate both sections separately, drop any negative signs and add the two values together. For the portion below the *x*-axis

$$\int_{-2}^{-1} x^3+1\,dx = \left[\frac{x^4}{4}+x\right]_{-2}^{-1} = \left(\frac{(-1)^4}{4}-1\right)-\left(\frac{(-2)^4}{4}-2\right) = -2\frac{3}{4}$$

The area represented by $\int_{-2}^{-1} x^3+1\,dx$, which lies below the *x*-axis, is $2\frac{3}{4}$ sq. units. For the portion above the *x*-axis

$$\int_{-1}^{0} x^3+1\,dx = \left[\frac{x^4}{4}+x\right]_{-1}^{0} = 0-\left(\frac{(-1)^4}{4}-1\right) = \frac{3}{4}$$

The area represented by $\int_{-1}^{0} x^3+1\,dx$, which lies *above* the *x*-axis, is $\frac{3}{4}$ sq. units. So the area of *R* is the sum of these two areas; that is $3\frac{1}{2}$ sq. units.

The general point is that when finding areas it is necessary to make a sketch first. If we make a sketch we will know in advance whether an area lies above or below the *x*-axis, so we can remove the negative sign at the first step. For example, we can split the integral into two and can place a minus (–) in front of a negative integral so as to make the area positive

$$R = -\int_{-2}^{-1} x^3 + 1 \, dx + \int_{-1}^{0} x^3 + 1 \, dx$$

Here the minus sign (–) appears before $\int_{-2}^{-1} x^3 + 1 \, dx$ automatically cancelling out the negative sign of the integral and resulting in the areas of the two portions of *R* being added together.

Example (4)

Sketch the graph of the curve $y = x^3 - 4x$.

(a) Find the area under this curve between the points 0 and 2;

(b) The definite integral between $x = -2$ and $x = 2$

(c) The total area bound by the curve and the *x*-axis between $x = -2$ and $x = 2$

(d) The area between 2 and 4.

Solution

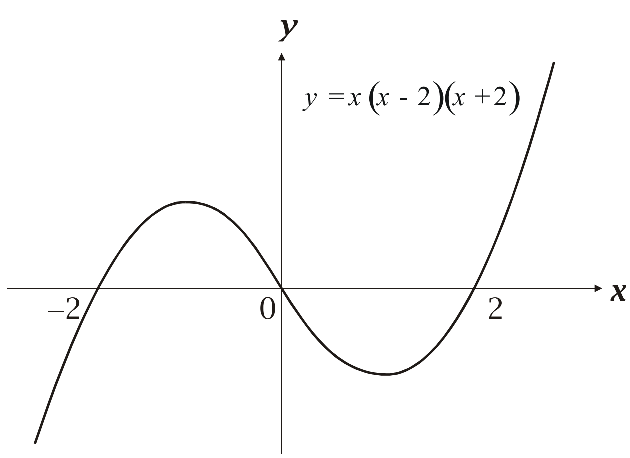
The points where $y = x^3 - 4x$ crosses the *x*-axis are where

$x^3 - 4x = 0$

$x(x-2)(x+2) = 0$

$x = -2, \; 0, \; 2$

From this we can obtain a sketch.

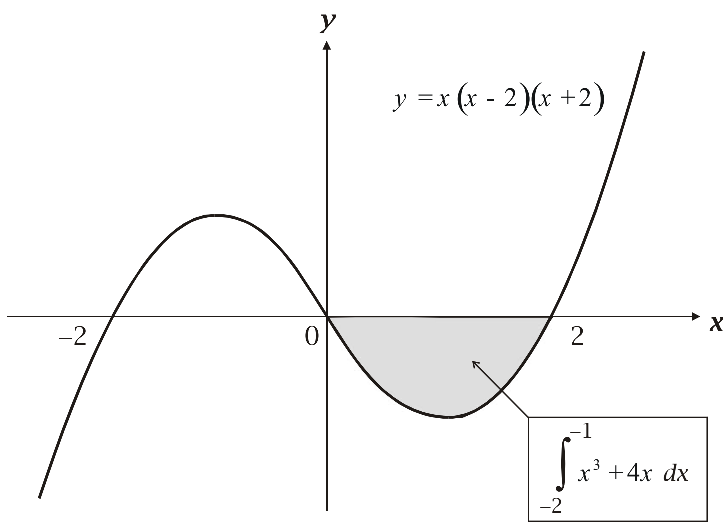


(*a*) The area between the points $x = 0$ and $x = 2$ lies below the *x*-axis, so the integral is negative. Consequently,

$$\text{Area} = -\int_0^2 x^3 - 4x\,dx$$

$$= -\left[\frac{1}{4}x^4 - 2x^2\right]_0^2 = -\{(4-8)-(0)\} = +4 \text{ sq. units}$$

We have removed the negative sign of the integral by adding a minus sign in front of it right at the beginning. This is possible because we have been asked to find the area and not the integral as such.



(*b*) We are asked to find the definite integral between –2 and 2.

$$I = \int_{-2}^2 x^3 - 4x\,dx$$

$$= \left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^2 = (4-8)-(4-8) = 0$$

The integral is zero because the area above the curve between $x = 0$ and $x = 2$ is equal in size to the area below the curve between $x = -2$ and $x = 0$. The integral is a signed quantity, so the two areas cancel out.

(*c*) To the actual area we have to split the integral into two parts; the integral between –2 and 0 and the integral between 0 and 2. We can ignore the signs and add both pieces together.

$$\text{Area} = \left|\int_{-2}^0 x^3 - 4x\,dx\right| + \left|\int_0^2 x^3 - 4x\,dx\right|$$

$$= 4 + 4$$

$$= 8$$

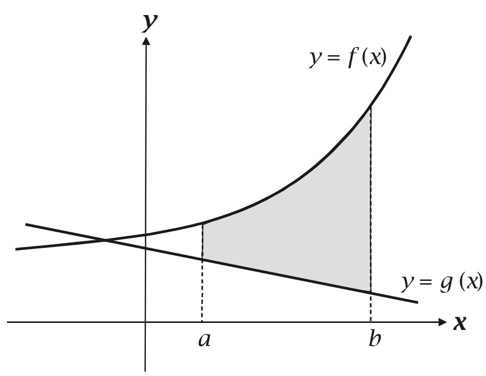
In this we use the modulus sign to cancel out any negative signs arising from evaluating an integral.

(*d*) To find the area between $x = 2$ and $x = 4$. The original sketch indicates that this are lies entirely above the *x*-axis so will be a positive quantity.

$$\text{Area} = \int_2^4 x^3 - 4x dx$$
$$= \left[\frac{1}{4}x^4 - 2x^2\right]_2^4$$
$$= (64 - 32) - (4 - 8) = 32 + 4 = 36$$

Area Bounded by Two Curves

Suppose we have two functions $f(x)$ and $g(x)$ with graphs looking something like this.

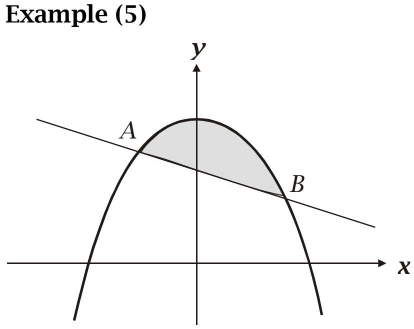


We are asked to find the area bounded by these two functions with limits a and b. Then the area is given by

$$\text{Area} = \int_a^b \left[f(x) - g(x)\right].dx$$

That is, in this case, we simply subtract one function from the other.

Example (5)



The diagram shows a sketch of the curve $y = 25 - x^2$ and the line $y = -x + 13$. The line and curve intersect at the points A and B.

(a) Find the coordinates of A and B.

(b) Find the area of the shaded region.

Solution

(*a*) We need to solve simultaneously

$$y = 25 - x^2 \qquad (1)$$
$$y = -x + 13 \qquad (2)$$

Substituting (1) in (2)

$$-x + 13 = 25 - x^2$$
$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$
$$x = 4 \text{ or } x = -3$$
$$x = 4 \quad \Rightarrow \quad y = 9$$
$$x = -3 \quad \Rightarrow \quad y = 16$$

(*b*) We are required to find

$$\int_{-3}^{4} 25 - x^2 - (-x+13)\,dx = \int_{-3}^{4} -x^2 + x + 12\,dx$$
$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 12x \right]_{-3}^{4}$$
$$= \left(-\frac{64}{3} + 8 + 48 \right) - \left(9 + \frac{9}{2} - 36 \right) = 57\frac{1}{6}$$

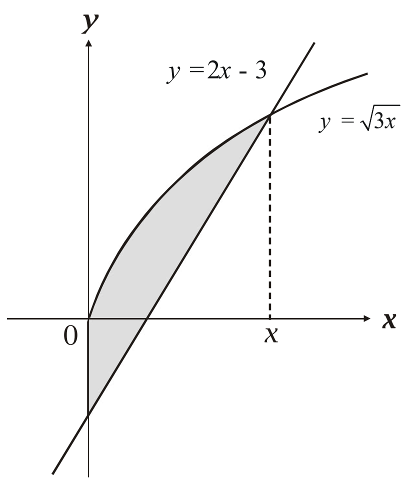
Example (6)

(*a*) Sketch the area bounded by the curves $y = \sqrt{3x}$ and $y = 2x - 3$ and the *y*-axis.

(*b*) Find the exact area.

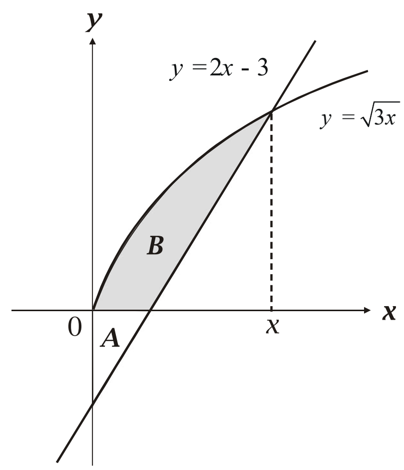
Solution

(*a*)

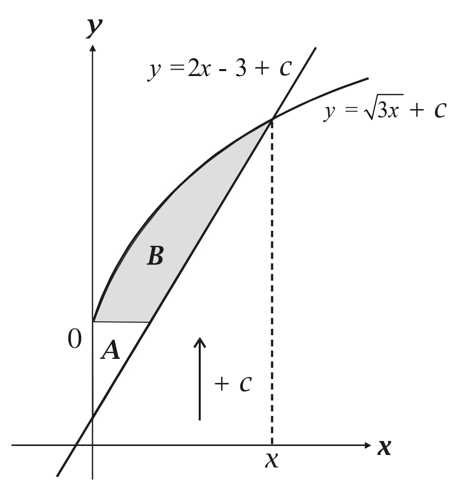


As the sketch indicates $y = \sqrt{3x}$ is not defined for negative *x* and to make it one-one the portion below the *x*-axis has been "cut off".

(*b*) To find the area of the shaded part. At first this looks problematic because there is a part of the area lying below the *x*-axis. This suggests we may have to cut the area up into separate segments and integrate each accordingly.



Fortunately, we do not have to do this. The formula $\text{Area} = \int_a^b \left[f(x) - g(x)\right].dx$ suffices. We just subtract one function from the other. To explain why this works, imagine that we move (translate) the whole figure up the "page" so that no part of the region of integration is lying below the *x*-axis.



This is equivalent to adding a constant c to both $f(x)$ and $g(x)$; so the expression for the area becomes

$\text{Area} = \int_a^b \left[f(x) + c - \left(g(x) + c\right)\right].dx$

The constant part just cancels out, to give

$$\text{Area} = \int_a^b \left[f(x) - g(x) \right] .dx$$

as before. So there is no requirement in this case to divide up the region. Now to solve this problem we need first to find the point of intersection of the two functions. We do this by equating them.

$$\sqrt{3x} = 2x - 3$$
$$3x = (2x-3)^2 = 4x^2 - 12x + 9$$
$$4x^2 - 15x + 9 = 0$$
$$(4x-3)(x-3) = 0$$
$$x = 3/4 \text{ or } x = 3$$

When $x = 3/4$, $y = \sqrt{9/4} = 3/2$
When $x = 3$, $y = 3$

The $x = 3/4$ solution corresponds to the intersection of the two curves below the *x*-axis. In the context, this solution has no meaning, so we discard it. Therefore, we require the integral

$$\begin{aligned}\text{Area} &= \int_0^3 \left[(3x)^{\frac{1}{2}} - (2x-3) \right] .dx \\ &= \int_0^3 \left[(3x)^{\frac{1}{2}} - 2x + 3 \right] .dx \\ &= \left[\tfrac{2}{9}(3x)^{\frac{3}{2}} - x^2 + 3x \right]_0^3 \\ &= \left(\frac{2 \times 27}{9} - 9 + 9 \right) - (0) \\ &= 6\end{aligned}$$

Exact and approximate integrals

Integration provides the exact solution to the problem of finding an area under a curve. There are approximate, that is, numerical methods as well for finding areas. For example, you should already be familiar with the trapezium method. Exact integrals for every function do not exist. Therefore, at some point if we are to find an answer to a given question involving an integral, we will have to use an approximate, that is, numerical method. Therefore, we must start learning about them from the beginning. Additionally, the theory of integration is based on the theory of numerical methods and not the other way around. To understand the concepts lying behind integration requires the study numerical methods.

Example (7)

(*a*) Find $\int_{0.5}^{1} \frac{1}{x^2} dx$

(*b*) Using the trapezium method with five intervals of width 0.1 find an approximation to

$\int_{0.5}^{1} \frac{1}{x^2} dx$ giving your answer to 4 decimal places.

(*c*) Find the absolute error and relative error in using the numerical approximation obtained in (ii) as an approximation to $\int_{0.5}^{1} \frac{1}{x^2} dx$.

Solution

(*a*) $\int_{0.5}^{1} \frac{1}{x^2} dx = \int_{0.5}^{1} x^{-2} dx = \left[-x^{-1}\right]_{0.5}^{1} = \left[-\frac{1}{x}\right]_{0.5}^{1} = -1 - \left(-\frac{1}{0.5}\right) = -1 + 2 = 1$

(*b*) The trapezium method is given by

$$A \approx \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right\}$$

Computing the ordinates

$x_0 = 0.5 \quad y_0 = y(0.5) = \frac{1}{0.5^2} = 4$

$x_1 = 0.6 \quad y_1 = y(0.6) = \frac{1}{0.6^2} = 2.77777...$

$x_2 = 0.7 \quad y_2 = y(0.7) = \frac{1}{0.7^2} = 2.040816...$

$x_3 = 0.8 \quad y_3 = y(0.8) = \frac{1}{0.8^2} = 1.5625$

$x_4 = 0.9 \quad y_4 = y(0.9) = \frac{1}{0.9^2} = 1.234567...$

$x_5 = 1 \quad y_5 = y(1) = \frac{1}{1^2} = 1$

The interval width is $h = 0.1$. Substituting into

$$A \approx \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right\}$$

$$A \approx \frac{0.1}{2}\left\{(4 + 1) + 2(2.777777 + 2.040816 + 1.5625 + 1.234567)\right\}$$
$$= 1.01156...$$
$$= 1.0116 \text{ (4.d.p.)}$$

(*c*) The absolute error is the size of the real value less the approximation.

$\text{absolute error} = |1 - 1.0116| = 0.0116$

$\text{relative error} = \frac{\text{absolute error}}{\text{real value}} \times 100\% = \frac{0.0116}{1} \times 100 = 1.16\%$