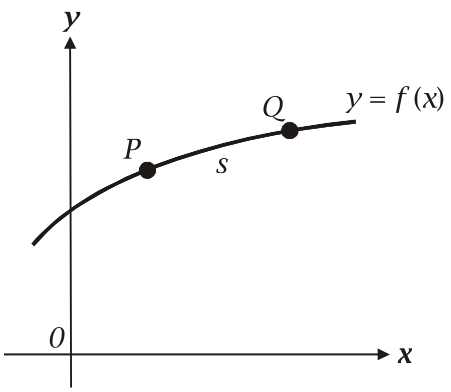
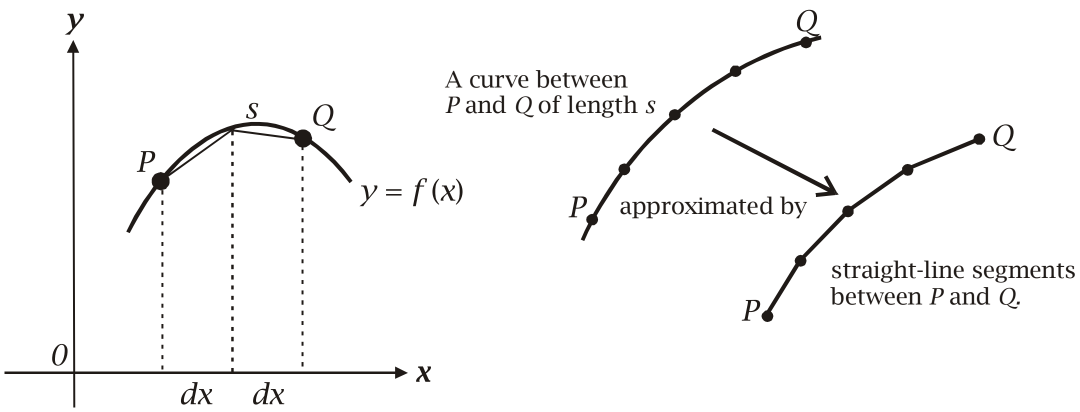
Arc length of a curve in Cartesian coordinates

Approximating the length of a curve

The problem under investigation here is to find the length of a curve between two points (say *P* and *Q*). The curve is given in terms of a function $y = f(x)$.

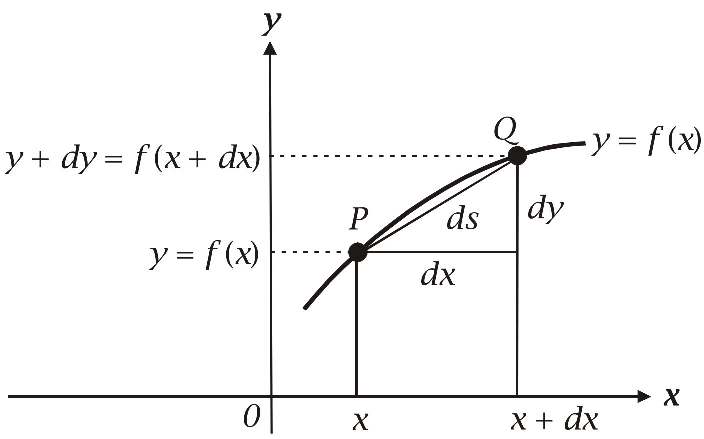


We will designate the required distance *s*. As with differential calculus, we will arrive at an exact expression for *s* by using the theory of closer and closer approximations.



We will approximate the length of the curve by adding up the straight-line segments that join points between *P* and *Q*. As we increase the number of segments the approximation gets better and better.

Suppose the length of one of these straight-line segments is designated *ds* and the corresponding small changes in the *x* and *y* directions are designated *dx* and *dy* respectively.



Then, as the above diagram indicates

$(ds)^2 = (dx)^2 + (dy)^2$

and

$ds = \sqrt{(dx)^2 + (dy)^2}$

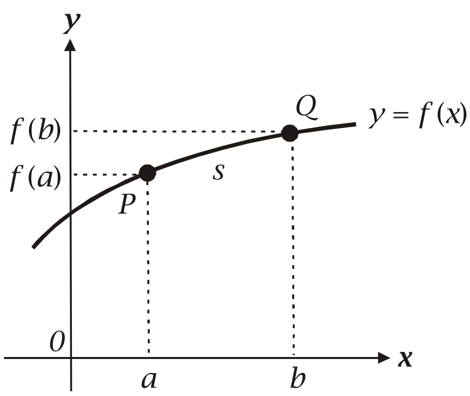
Then the approximate length along the curve is the sum of the straight-line segments

$s = \sum ds$

Hence, the exact curve is

$s = \lim_{\delta x \to 0} \sum ds$

It is the limit of the sum of the small increments along the curve from *P* to *Q* as the length of the increment gets smaller and smaller. Also the point *P* is given by, say, $y = f(a)$ and the point *Q* by $y = f(b)$.



$$s = \lim_{dx \to 0} \sum_{a}^{b} \sqrt{(dx)^2 + (dy)^2} = \int_{a}^{b} \sqrt{(dx)^2 + (dy)^2}$$

This formula is not in a very manageable form. However, we are allowed to manipulate the terms *dx* and *dy* as if they were functions, which in fact they are – they are called *differentials*. This enables us to write the arc length in the following more convenient form.

$$\begin{aligned} s &= \int_{a}^{b} \sqrt{(dx)^2 + (dy)^2} \\ &= \int_{a}^{b} \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)} \\ &= \int_{a}^{b} \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx \end{aligned}$$

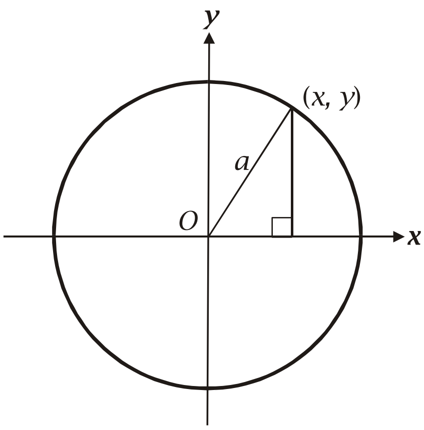
This computes the arc length in terms of the variable x¸ but we could also write it in terms of y.

$$s = \int_{f(a)}^{f(b)} \left(\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right) dy$$

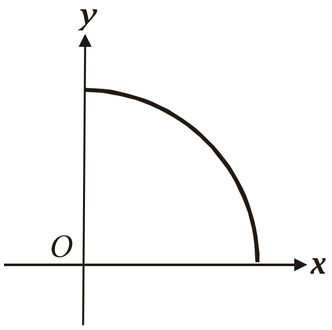
We will now proceed to illustrate the application of this formula.

The circumference of the circle

Suppose we wish to find the circumference of a circle radius a.



A circle of radius a has an equation $x^2 + y^2 = a^2$. Since there are two y values corresponding to a given x value, this is not a function. This makes integrating it difficult. However, the length of the circle can be treated as four times the length of one of the quadrants.



On this segment it is possible to make a one-one function. Here we may write $y = \sqrt{a^2 - x^2}$ with the convention that we take only the positive root. Then the arc length of one segment is given by $\int_0^a \left(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right) dx$. This formula requires us to find $\frac{dy}{dx}$. To do so

$$y = \left(a^2 - x^2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(a^2 - x^2\right)^{-\frac{1}{2}} \times -2x = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{a^2 - x^2}$$

Then on substituting into the formula

$$\begin{aligned}
s &= \int_0^a \left(\sqrt{1 - \left(\frac{dy}{dx} \right)^2} \right) dx \\
&= \int_0^a \left(\sqrt{1 + \left(\frac{x^2}{a^2 - x^2} \right)} \right) dx \\
&= \int_0^a \left(\sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}} \right) dx \\
&= \int_0^a \left(\sqrt{\frac{a^2}{a^2 - x^2}} \right) dx \\
&= a\int_0^a \left(\frac{1}{\sqrt{a^2 - x^2}} \right) dx \\
&= a\left[\sin^{-1}\left(\frac{x}{a} \right) \right]_0^a \\
&= a\left[\sin^{-1}(1) - 0 \right] \\
&= a\left(\frac{\pi}{2} - 0 \right) \\
&= \frac{\pi a}{2}
\end{aligned}$$

This gives the arc length of one quarter of the circumference of the circle. Hence the full circumference is

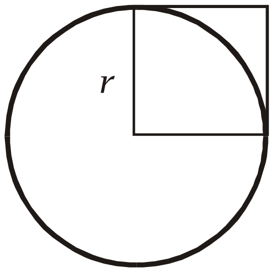
$$C = 4 \times \frac{\pi a}{2} = 2\pi a$$

This is the familiar formula for the circumference of the circle, which we have now proven. You may not have been aware that $C = 2\pi a$ is actually a theorem – you have been so familiar with it as a result! Logically, the formula for area comes first

$$A = \pi r^2$$

This is not a theorem but actually a definition – it is the definition of π (pi) as the ratio of the area of the circle (*A*) to the area of the square of its radius

$$\pi = \frac{A}{r^2}$$



That is π (pi) is the ratio of the area of a circle of radius *r* to the area of a square that has a side of length *r*. Having defined π in this way, it is then that we prove the formula for the circumference by means the formula, $\int_0^a \left(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right) dx$, for the arc-length of a curve in Cartesian coordinates that we have just introduced.

So from the definition of π as $\pi = \frac{A}{r^2}$ we prove that the circumference of a circle is $C = 2\pi r$.

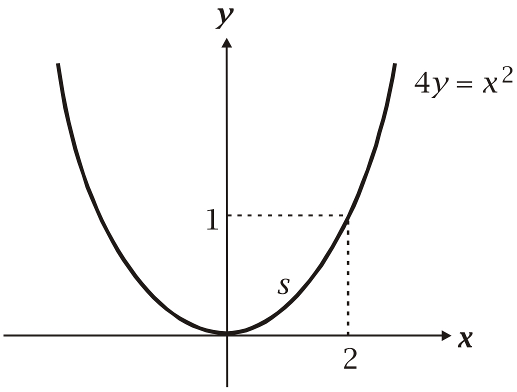
This leads us into the question of the properties of π, which is a fascinating study. The question that haunted the Greeks was whether the area of the circle could be constructed from the area of the square using only a compass and ruler. In the C19th Galois showed that this was not possible, which meant that π could not be a rational number. Later Lindenbaum showed that π is a *transcendental* number, which means that it is not the root of any polynomial function with rational coefficients

Further examples

Example (1)

Find the arc length of a parabola $x^2 = 4y$ from the vertex to $x = 2$

Solution



We have $y = \frac{x^2}{4}$ which is a parabola. Hence

$$\frac{dy}{dx} = \frac{x}{2}$$

The arc length formula is

$$s = \int_a^b \left(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right) dx$$

and on substitution we get

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + \left(\frac{x}{2} \right)^2} dx \\ &= \int_0^2 \sqrt{1 - \frac{x^2}{4}} \ dx \\ &= \int_0^2 \sqrt{\frac{4 + x^2}{4}} \ dx \\ &= \frac{1}{2} \int_0^2 \sqrt{4 + x^2} \ dx \end{aligned}$$

The integral of this expression is a standard result found by a hyperbolic substitution (with which you should be familiar at this stage) and is given by

$$s = \frac{1}{2} \left[x\sqrt{x^2 + 4} + 2\sinh^{-1}\left(\frac{x}{2} \right) \right]$$

This is found by substitution into the formula $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$.

The result also can be given in logarithmic form.

$$
\begin{aligned}
s &= \frac{1}{2}\left[x\sqrt{x^2+4} + 2\ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)\right]_0^2 \\
&= \frac{1}{2}\left[\frac{1}{2}\times 2\sqrt{8} + 2\left\{\ln\left(2+\sqrt{8}\right) - \ln 2\right\}\right] \\
&= \sqrt{2} + \ln\left(\frac{2+2\sqrt{2}}{2}\right) \\
&= \sqrt{2} + \ln\left(1+\sqrt{2}\right) \\
&= 2.296 \ \ (3\text{ s.f.})
\end{aligned}
$$

Example (2)

(i) Prove that the length of the arc of the curve $y = \log_2(x)$ between the points where $x = 2$ and $x = 3$ is given by $s = \int_2^3 \sqrt{1 + \frac{1}{x^2 \ln^2(x)}}\,dx$

(ii) Use the substitution $\sinh^2(t) = \frac{1}{x^2\ln^2(2)}$ to evaluate this integral to 3 s.f.

Solution

(i) $y(x) = \log_2(x) = \frac{\ln(x)}{\ln(2)}$

$$\frac{dy}{dx} = \frac{1}{x\ln(2)}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{x^2\ln^2(2)}$$

On substitution into the formula $s = \int_a^b \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) dx$.

$$s = \int_2^3 \sqrt{1 + \frac{1}{x^2\ln^2(x)}}\,dx$$

Proven.

(ii) $\dfrac{1}{x^2\ln^2(2)} = \sinh^2(t)$

$$x = \frac{1}{\ln(2)\sinh(t)}$$

$$dx = -\frac{\cosh(t)}{\ln(2)\sinh^2(x)}\ dt$$

$$t = \sinh^{-1}\left(\frac{1}{x\ln(2)}\right)$$

$$s = \int_2^3 \sqrt{1+\frac{1}{x^2\ln^2(x)}}\ dx$$

$$= \int_{\sinh^{-1}\left(\frac{1}{2\ln(2)}\right)}^{\sinh^{-1}\left(\frac{1}{3\ln(2)}\right)} \sqrt{1+\sinh^2 t}\left(-\frac{\cosh(t)}{\ln(2)\sinh^2(x)}\right)dt$$

$$= \int_{\sinh^{-1}\left(\frac{1}{2\ln(2)}\right)}^{\sinh^{-1}\left(\frac{1}{3\ln(2)}\right)} \left(-\frac{\cosh^2(t)}{\ln(2)\sinh^2(t)}\right)dt$$

$$= -\frac{1}{\ln(2)}\int_{\sinh^{-1}\left(\frac{1}{2\ln(2)}\right)}^{\sinh^{-1}\left(\frac{1}{3\ln(2)}\right)} \left(\frac{1+\sinh^2(t)}{\sinh^2(t)}\right)dt$$

$$= -\frac{1}{\ln(2)}\int_{\sinh^{-1}\left(\frac{1}{2\ln(2)}\right)}^{\sinh^{-1}\left(\frac{1}{3\ln(2)}\right)} \left(\operatorname{cosech}^2(t)+1\right)dt$$

$$= \frac{1}{\ln(2)}\Big[\coth(t)-t\Big]_{\sinh^{-1}\left(\frac{1}{2\ln(2)}\right)}^{\sinh^{-1}\left(\frac{1}{3\ln(2)}\right)} \qquad \left[\frac{d}{dt}\coth(t) = -\operatorname{cosech}^2(t)\right]$$

$$= \frac{1}{\ln(2)}\Big[\sqrt{1+\operatorname{cosech}^2(t)}-t\Big]_{\sinh^{-1}\left(\frac{1}{2\ln(2)}\right)}^{\sinh^{-1}\left(\frac{1}{3\ln(2)}\right)} \qquad \left[\coth^2(t) = \operatorname{cosech}^2(t)+1\right]$$

$$= \frac{1}{\ln(2)}\left[x\ln(2)\sqrt{1+\frac{1}{x^2\ln^2(2)}}-\sinh^{-1}\left(\frac{1}{x\ln(2)}\right)\right]_2^3 \qquad \left[\frac{1}{x^2\ln^2(2)} = \sinh^2(t) \Rightarrow \operatorname{cosech}^2(t) = x^2\ln^2(2)\right]$$

$$= \frac{1}{\ln(2)}\left[\sqrt{1+x^2\ln^2(2)}-\sinh^{-1}\left(\frac{1}{x\ln(2)}\right)\right]_2^3$$

$$= 1.16\ (3.s.f.)$$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