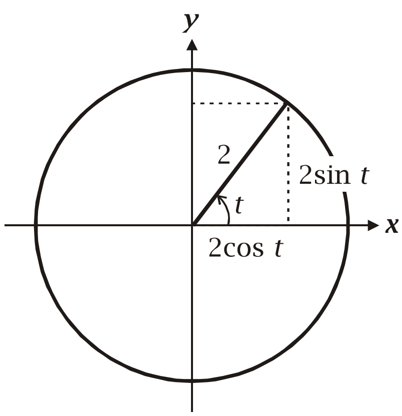
Arc length of a curve: Parametric form

Parametric equations

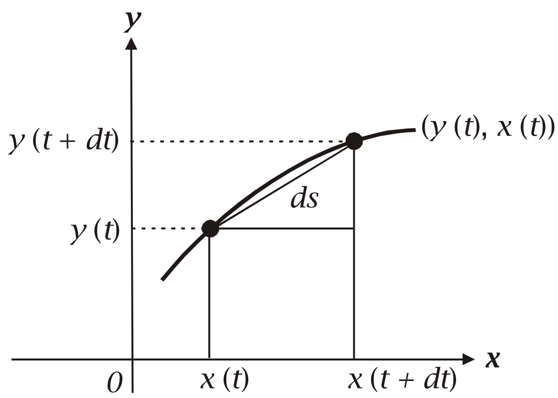
The expression $y = ax^2$ describes a parabola as a relationship between the Cartesian coordinates x and y. The expression $r = a(1-\cos\theta)$ describes a cardioid as a relationship between polar coordinates r and θ. It is often more convenient to describe a curve in parametric form. In parametric form we specify the coordinates of a point on the curve by separate functions of another variable, called a parameter: $\alpha(t) = (x(t), y(t))$. $\alpha(t)$ is the position function. For example, a circle of radius 2 is given in parametric form by $\alpha(t) = (2\cos t, 2\sin t)$



When $t = \pi/4$ the coordinates are $\left(2\sin\frac{\pi}{4}, \cos\frac{\pi}{4}\right) = \left(\sqrt{2}/2, \sqrt{2}/2\right)$.

Arc length in parametric form

We now need to derive the formula for the arc length when the equation is given in parametric form. We approximate the length of the curve by line segments. As the parameter increases by dt to $t + dt$ the x-coordinate changes from $x(t)$ to $x(t+dt)$ and the y-coordinate from $y(t)$ to $y(t+dt)$.



The line segment is

$$ds = \sqrt{\left[x(t+dt)-x(t)\right]^2 + \left[y(t+dt)-y(t)\right]^2}$$
$$= \left\{\sqrt{\left(\frac{x(t+dt)-x(t)}{dt}\right)^2 + \left(\frac{y(t-dt)-y(t)}{dt}\right)^2}\right\}dt$$

In the limit as $dt \to 0$ we have $\frac{dx}{dt} = \lim_{dt\to 0}\left\{\frac{x(t-dt)-x(t)}{dt}\right\}$ and likewise $\frac{dy}{dt} = \lim_{dt\to 0}\left\{\frac{y(t-dt)-y(t)}{dt}\right\}$.

Hence $ds = \left\{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right\} dt$. But the curve is approximated by the sum of the line segment.

Hence, the arc length, s, is given by $s = \int_{t_1}^{t_2}\left\{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right\} dt$.

Examples

Example (1)

Find the length of the circumference of a circle of radius a using the parametric form.

Solution

Here θ represent the parameter standing for the angle subtended at the centre of the circle. Then the circle has parametric form

$$\alpha(\theta) = (a\cos\theta, a\sin\theta)$$

where a is the radius.

$$x(\theta) = a\cos\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta$$
$$y(\theta) = a\sin\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = a\cos\theta$$

$$\begin{aligned}
s &= \int_0^{2\pi} \left\{ \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \right\} \cdot d\theta \\
&= \int_0^{2\pi} \left(\sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2} \right) d\theta \\
&= \int_0^{2\pi} \left(a\sqrt{\sin^2\theta + \cos^2\theta} \right) d\theta \\
&= a\int_0^{2\pi} d\theta \\
&= a[\theta]_0^{2\pi} \\
&= 2\pi a
\end{aligned}$$

Example (2)

Find the length of $x = 4t^2 + 6t, \quad y = 3t^2 - 8t \quad 1 \le t \le 3$.

Solution

$$\frac{dx}{dt} = 8t + 6 \qquad \frac{dy}{dt} = 6t - 8$$

$$\begin{aligned}
s &= \int_{t_1}^{t_2} \left\{ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right\} dt \\
&= \int_1^3 \sqrt{64t^2 + 48t + 36 + 36t^2 - 48t + 64}\ dt \\
&= \int_1^3 \sqrt{100t^2 + 100}\ dt \\
&= 10\int_1^3 \sqrt{1 + t^2}\ dt \\
&= 10\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left|t + \sqrt{1+t^2}\right| \right]_1^3 \\
&= 10\left(\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln\left(3 + \sqrt{10}\right) - \frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(1 + \sqrt{2}\right) \right) \\
&= 45.048 \\
&= 45.0\ (3.s.f.)
\end{aligned}$$