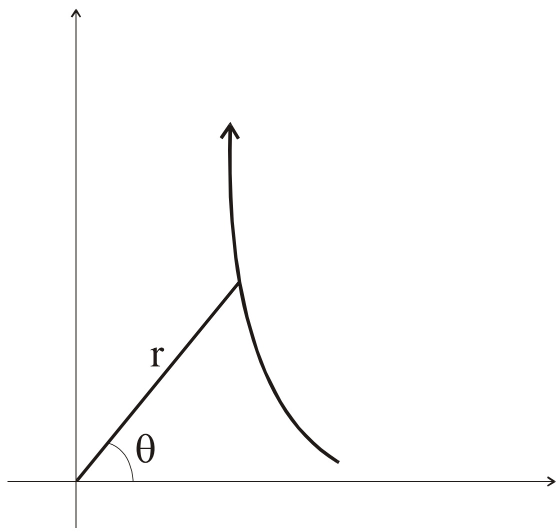
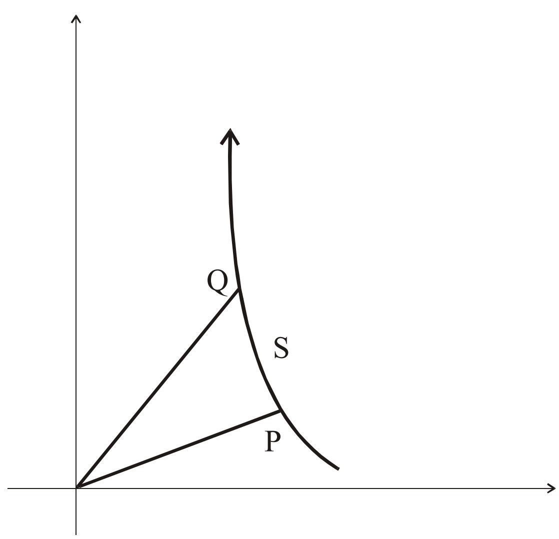
Arc length of a curve – in Polar coordinates

Some curves are more conveniently given in polar form – that is, as functions of the angle swept out from the x-axis (in an anti-clockwise direction) and the distance from the origin.



The function is expressed as a relationship between r and θ.

Our task is to find the length of this curve between two points P and Q.



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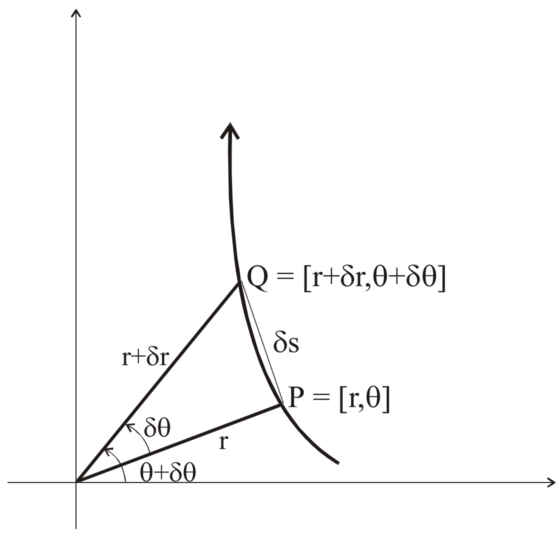
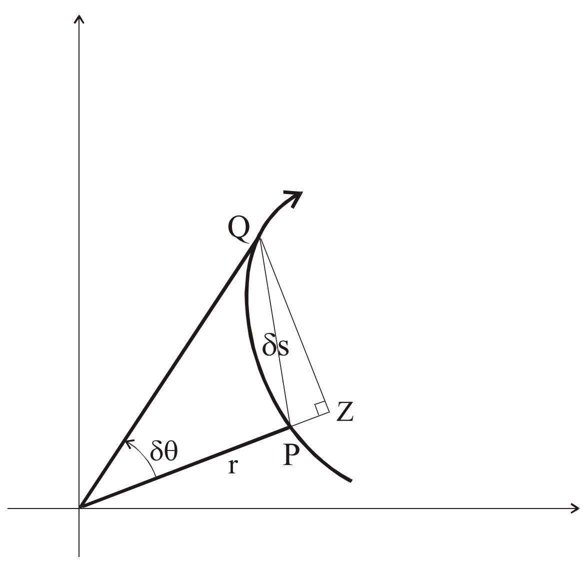
We derive this formula in the usual way by dividing the curve into segments and approximating each segment by a straight line.

Consider one such argument of length δs.



As the angle θ is increased by $\delta\theta$, the distance of the curve from the origin changes by δr from r to $r+\delta r$.

To find an expression for δs we construct a right-angled triangle, thus



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Then $\delta s = \sqrt{(PZ)^2 + (ZQ)^2}$

Now PQ is approximately equal to the arc length given by $r\delta\theta$.

This approximation would be exact if we took the limit; that is $\delta\theta \to 0$.

The length PZ is the change in the r coordinate, i.e. $PZ = \partial r$. Hence,

$$\delta s = \sqrt{(r\delta\theta)^2 - (\delta\theta)^2}$$

And in the limit the "infinitesimal" increase in arclength, *ds*, is given by:
You should remove the reference to infinitessimal.

$$ds = \sqrt{r^2 (d\theta)^2 + (dr)^2}$$
$$= \left\{ \sqrt{\frac{r^2 (d\theta)^2}{(d\theta)^2} + \frac{(dr)^2}{(d\theta)^2}} \right\} \cdot d\theta$$
$$= \left\{ \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right\} \cdot d\theta$$

The arclength is given by the sum of small segments ∂s. Hence,

$$s \approx \sum_{\theta_1}^{\theta_2} \delta s$$

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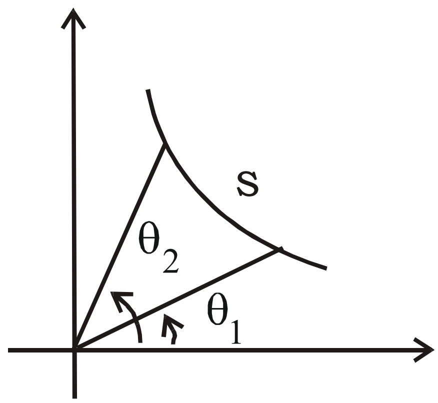

Hence: $s = \int_{\theta_1}^{\theta_2} \left\{ \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right\} \cdot d\theta$

This is the arclength formula in polar coordinates.

<u>Example</u>

Find the length of the cardioid with equation
$r = a(1 - \cos\theta)$

Solution

We visualise the cardioid by the usual technique of evaluating points and sketching

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\cos\theta$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1
$1-\cos\theta$	0	$\sqrt{2}-1/\sqrt{2}$	1	$1+\sqrt{2}/\sqrt{2}$	2	$1+\sqrt{2}/\sqrt{2}$	1	$\sqrt{2}-1/\sqrt{2}$	0

This gives us the cardioid:-

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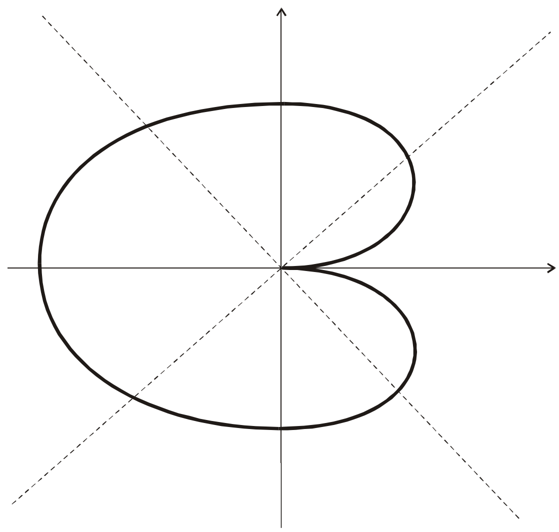

We are finding the line length as the angle sweeps out from 0 to 2π.

$$r = a\left(1-\cos\theta\right)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$s = \int_0^{2\pi} \left\{ \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right\} \cdot d\theta$$

$$= \int_0^{2\pi} \sqrt{\left(a\left(1-\cos\theta\right)\right)^2 + \left(a\sin\theta\right)^2} \cdot d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2\left(1-2\cos\theta+\cos^2\theta\right)+a^2\sin^2\theta} \cdot d\theta$$

$$= a\int_0^{2\pi} \sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta} \cdot d\theta$$

$$= a\int_0^{2\pi} \sqrt{2-2\cos\theta} \cdot d\theta$$

We recall that since $\cos 2A = \cos^2 A - \sin^2 A = 1-2\sin^2 A$

that $\cos\theta = 1-2\sin^2\left(\theta/2\right)$

Therefore

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$$s = a\int_0^{2\pi} \sqrt{2-2\left(1-2\sin^2\left(\theta/2\right)\right)} \cdot d\theta$$

$$= a\int_0^{2\pi} \sqrt{4\sin^2\left(\theta/2\right)} \cdot d\theta$$

$$= 2a\int_0^{2\pi} \sin\left(\theta/2\right) \cdot d\theta$$

$$= 2a\left[-2\cos\left(\theta/2\right)\right]_0^{2\pi}$$

$$= 4a\left(-\cos\pi + \cos 0\right)$$

$$= 4a\left(1+1\right) = 8a$$

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