Arithmetic Progressions

Sequences and series

A *sequence* is any string of numbers in a given order.

Example (1)

Consider the following sequence of numbers

1 4 7 10 13 ...

(*a*) Complete the following mapping diagram for this sequence.

$$\begin{array}{lll} 1 & \to & 1 \\ 2 & \to & 4 \\ 3 & \to & \ldots \\ 4 & \to & \ldots \\ 5 & \to & \ldots \\ n & \to & \ldots \end{array}$$

(*b*) Explain in words how you would continue this sequence.

Solution

(*a*)

$$\begin{array}{lll} 1 & \to & 1 \\ 2 & \to & 4 \\ 3 & \to & 7 \\ 4 & \to & 10 \\ 5 & \to & 13 \\ n & \to & 1+3(n-1)=3n-2 \end{array}$$

(*b*) Each number is separated from the next by the difference of 3. Therefore, to continue the sequence, we keep adding 3 to each successive term. This kind of sequence of numbers, where each successive term is separated from the next by a *common difference*, is called an *arithmetic progression.*

A *series* is a special type of sequence. A series is formed from an existing sequence by the addition of succeeding terms.

Example (2)

In the preceding example we saw the following arithmetic progression

1 4 7 10 13 ...

Using these numbers create a *series* by completing the following.

1	$\rightarrow$	1
2	$\rightarrow$	$1+4=5$
3	$\rightarrow$	$1+4+7=12$
4	$\rightarrow$	$1+4+7+10=$....
5	$\rightarrow$	...
6	$\rightarrow$	...

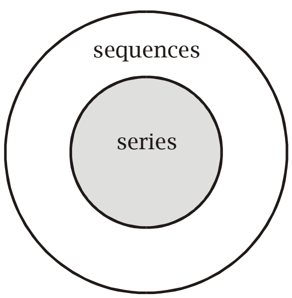
Solution

1	$\rightarrow$	1
2	$\rightarrow$	$1+4=5$
3	$\rightarrow$	$1+4+7=12$
4	$\rightarrow$	$1+4+7+10=22$
5	$\rightarrow$	$1+4+7+10+13=35$
6	$\rightarrow$	$1+4+7+10+13+16=51$

The numbers generated in this example

1 5 12 22 35 51 ...

form an example of a series. All series are sequences, but not all sequences are series.



Compare the following

Arithmetic progression (A sequence)			Sum of successive terms (A series)		
n		u_n	n		S_n
1	$\rightarrow$	1	1	$\rightarrow$	1
2	$\rightarrow$	4	2	$\rightarrow$	5
3	$\rightarrow$	7	3	$\rightarrow$	12
4	$\rightarrow$	10	4	$\rightarrow$	22
5	$\rightarrow$	13	5	$\rightarrow$	35
n	$\rightarrow$	$3n-2$	n	$\rightarrow$	?

In this diagram we have introduced a little extra notation. We use u_n to represent the nth term of the arithmetic progression, and S_n to represent the nth term of the correspond series formed by summing up successive terms of the arithmetic progression. Here

$u_1 = 1 \quad S_1 = 1$
$u_2 = 4 \quad S_2 = 5$
$u_3 = 7 \quad S_3 = 12$
$\ldots \quad \ldots$

Using this notation we can write

$$
\begin{aligned}
S_1 &= u_1 \\
S_2 &= u_1 + u_2 \\
S_3 &= u_1 + u_2 + u_3 \\
&\ldots \\
S_n &= u_1 + u_2 + u_3 + \ldots. + u_{n-1} + u_n
\end{aligned}
$$

This would be valid for any sequence and its corresponding series.

The sum of an arithmetic progression

An arithmetic progression is formed by progressively adding a number to the latest member of the sequence. The terms *arithmetic sequence* and *arithmetic progression* mean the same thing. The difference between successive terms is called the *common difference* and the starting term is called the *first term*. As we have seen above, in the sequence 1, 4, 7, 10 ... the first term is 1, and the common difference is 3. We introduced the symbol u_n to represent the *n*th term in an arithmetic progression. In the progression 1, 4, 7, 10... we have

$u_1 = 1 \quad u_2 = 4 \quad u_3 = 7 \quad u_4 = 10 \qquad u_n = 1 + 3(n-1) = 3n - 2$

For an arithmetic progression in general we let the first term = a and the common difference = d, and the arithmetic progression is

$$
\begin{aligned}
u_1 &= a \\
u_2 &= a + d \\
u_3 &= a + 2d \\
&\vdots \\
u_n &= a + (n-1)d
\end{aligned}
$$

Here, there are n terms, the first term is $u_1 = a$ and the last term is $u_n = a + (n-1)d$. In the progression 1, 4, 7, 10 ... we have

$a = 1 \quad d = 3$

Example (3)

In the arithmetic progression

$-2 \qquad 0.5 \qquad 3 \qquad 5.5 \qquad \ldots$

(*a*) Find the first term a and the common difference d.

(*b*) If there are 50 terms in the progression, what is the last term?

Solution

(a) $\quad a = -2$

$\quad\quad d = 2.5$

(b) $\quad u_{50} = a + (n-1)d$

$\quad\quad = -2 + 49 \times 2.5$

$\quad\quad = 120.5$

Arithmetic progressions arise in many real life situations. For example, consider a fixed interest account – starting with a certain amount, each month a fixed interest is added. This is an arithmetic progression. We would like to know how much money in total we would have after a certain number of months. In other words, we want to know the total or *sum* of an arithmetic progression. Of course, we could simply add up all the terms, but if there are a lot of them this could be tedious, and in any case, in situations such as these, mathematicians look for a *short cut* to the solution. So we are seeking a formula that gives the value of S_n in

$S_n = a + (a+d) + (a+2d) + ... + (a + (n-1)d)$.

First we will state the result and look at some examples. After that we will prove the result. The sum of an arithmetic progression is given by

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

In addition, it can also be shown that the sum of an arithmetic progression is given by

$$S_n = \frac{n}{2}\left(u_0 + u_{n-1}\right)$$

That is

$$S_n = \frac{n}{2}\left(\text{first term} + \text{last term}\right)$$

So there are two formulae and both are useful.

Example (4)

Find the 9th term of the arithmetic progression

1, 4, 7, 10

Find also

(*a*) The sum to 9 terms

(*b*) The sum to 20 terms.

Solution

The *n*th term of an arithmetic progression is $u_n = a + (n-1)d$. Here

$n = 9 \qquad a = 1 \qquad d = 3$

$$\begin{aligned} u_9 &= 1 + (9-1) \times 3 \\ &= 1 + 24 \\ &= 25 \end{aligned}$$

(*a*) The sum of an arithmetic progression is

$$S_n = \frac{n}{2}(\text{first term} + \text{last term})$$

$$S_9 = \frac{9}{2}(1 + 25) = 117$$

(*b*) We use the other formula

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_{20} = \frac{20}{2}(2 \times 1 + (20-1)3) = 590$$

The formula for the sum of an arithmetic progression is an example of a *short-cut* theorem. It provides a quick route from the progression to its sum, without having to go the long way of adding up all the terms.

Problems involving arithmetic progressions

Problems can be set in which you are asked to find an unknown term.

Example (5)

The general term of an arithmetic progression is given by $u_n = 5 + 3n$. Find the common difference and the first term.

Solution

Substituting $n = 1$ we get the first term

$a = u_1 = 8$

Substituting $n = 2$

$u_2 = 11$

The common difference is the difference between successive terms, so

common difference = $d = u_2 - u_1 = 8 - 5 = 3$

The next example requires the use of the quadratic formula to find an unknown quantity.

Example (6)

The sum of an arithmetic progression is –600. If the first term is 12, and the common difference is –3, how many terms are in the progression?

Solution

$S_n = -600 \qquad a = 12 \qquad d = -3$

The sum of an arithmetic progression is

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

On substitution into this formula

$$\begin{aligned}
-600 &= \frac{n}{2}\left(2\times 12 + (n-1)\times(-3)\right)\\
-1200 &= n(24 - 3n + 3)\\
-1200 &= 27n - 3n^2\\
3n^2 - 27n - 1200 &= 0
\end{aligned}$$

To find n we need to solve this quadratic equation. Using the quadratic formula

$$\begin{aligned}
n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&= \frac{27 \pm \sqrt{27^2 + 4\times 3\times 1200}}{6}\\
&= 25, \text{ or } -16
\end{aligned}$$

However, the negative solution is not possible, hence $n = 25$.

Problems leading to simultaneous equations are also commonly set.

Example (7)

In an arithmetic progression, the 9th term is three times that of the 3rd term, and the 12th term is 36. Find the common difference and the sum of the terms from the 3rd to the 12th inclusive.

Solution

The nth term of an arithmetic progression is

$$u_n = a + (n-1)d$$

Substitution into this formula for the fact that the twelfth term is 36 gives

$$a + 11d = 3$$

We are also told that the 9th term is three times that of the 3th term.

$$\begin{aligned}
u_9 &= 3u_3\\
a + 8d &= 3(a + 2d)
\end{aligned}$$

$$a + 8d = 3a + 6d$$
$$2a = 2d$$
$$a = d$$

We already showed that

$$u_{12} = a + 11d = 36$$

Substituting $a = d$ into this we get

$$d + 11d = 36$$
$$12d = 36$$
$$d = 3$$
$$a = 3$$

The sum of this arithmetic progression from the 3rd to the 12th term inclusive is $S_{12} - S_2$.

The formula for an arithmetic progression is

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

Thus

$$S_{12} = \frac{12}{2}\left(2 \times 3 + 11 \times 3\right) = 6 \times 39 = 234$$
$$S_2 = \frac{2}{2}\left(2 \times 3 + 1 \times 3\right) = 1 \times 9 = 9$$
$$S = S_{12} - S_2 = 234 - 9 = 225$$

Proof of the formula for the sum of an arithmetic progression

To prove that for an arithmetic progression with first term *a* and common difference *d*, the sum to *n* terms is given by

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

Proof

The sum to *n* terms of the arithmetic progression with first term *a* and common difference *d* is

$$S_n = u_1 + u_2 + u_3 + ... + u_{n-1} + u_n$$

That is

$$S_n = a + (a + d) + (a + 2d) + ... + (a + (n-1)d) \qquad (1)$$

Reversing the order of the terms in this then the sum is also given by

$$S_n = u_n + u_{n-1} + u_{n-2} + ... + u_2 + u_1$$

That is

$$S_n = (a + (n-1)d) + (a + (n-2)d) + \ldots + (a + d) + a \qquad (2)$$

This is the first sum just written backwards. Adding the two equations, (1) and (2), together on a term by term basis we get

$$2S_n = \left(2a + (n-1)d\right) + \left(2a + (n-1)d\right) + \ldots + \left(2a + (n-1)d\right)$$

To convince you of this, the first term in the progression written the "right way" around is

$$u_1 = a$$

And the first term in the progression written "back to front" is

$$u_n = a + (n-1)d$$

So when they are added together

$$u_1 + u_n = a + \left(a + (n-1)d\right) = 2a + (n-1)d$$

The same result is obtained whenever two pairs of terms in the two progressions are added together.

$$u_1 + u_{n-1} = 2a + (n-1)d$$
$$u_2 + u_{n-2} = 2a + (n-1)d$$

And so forth. Thus, when we add up all the terms on a term-by-term basis we obtain $2a + (n-1)d$ added to itself n times. Hence

$$2S_n = n\left(2a + (n-1)d\right)$$
$$\therefore S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

The key to being good at mathematics is to *learn the proofs!* Students who know the proofs have fewer difficulties with the problems than students who do not know the proofs.