Bayes Theorem

The definition of a conditional probability is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Bayes Theorem answers the question of "Given $P(A|B)$ what is $P(B|A)$?"

Bayes theorem provides the formula

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

<u>Example</u>

An Anthology of poetry is made of the work of three poets, Donne, Herbert and Marvell. 45% of the poems are written by Donne, 25% are by Herbert and the rest are by Marvell. The poems are either above love (or soul union) or religion (sin & redemption).

The probability that a poem by Donne will be religious is 0.5, and the respective probabilities for Herbert and Marvell are 0.9 and 0.4. What is the probability that a love poem chosen at random will be done by Donne?

Solution

Let

D be the event the event the poem is by Donne

H be the event the event the poem is by Herbert

M be the event the event the poem is by Mavell

X be the event the event the poem is a Love poem

$\overline{X}$ be the event the event the poem is a religious poem.

We are given

$P(D) = 0.45$

$P(H) = 0.25$

$P(M) = 1 - 0.45 - 0.25 = 0.30$

$P(\bar{X}|D) = 0.5$

$P(\bar{X}|H) = 0.9$

$P(\bar{X}|M) = 0.4$

We need to find $P(D|X)$

We use Bayes theorem to "Reverse the conditions" that is

$$P(D|X) = \frac{P(X|D) \times P(D)}{P(X)}$$

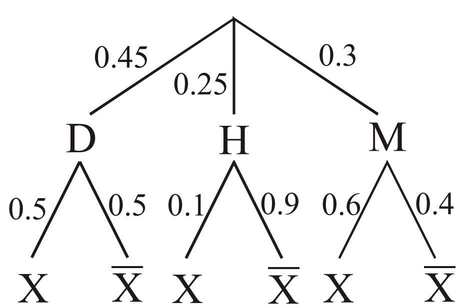
Now

$P(X|D) = 1 - P(\bar{X}|D) = 1 - 0.5 = 0.5$

We have $P(D) = 0.45$

And require $P(X)$

$P(X)$ is the total probability that the poem will be a love poem. To find this we construct a probability tree.



Therefore,

$$P(X) = (0.45 \times 0.5) + (0.25 \times 0.1) + (0.3 \times 0.6) = 0.43$$

Hence, or substituting into Bayes Theorem

$$\begin{aligned} P(D|X) &= \frac{P(X|D) \times P(D)}{P(X)} \\ &= \frac{0.5 \times 0.45}{0.43} \\ &= 0.523 (3.D.P.) \end{aligned}$$

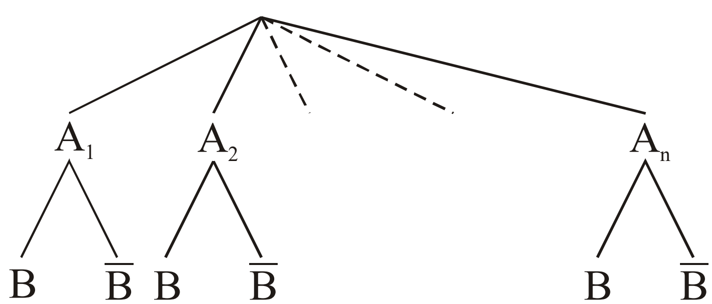
This is a relatively "straight-forward" substitution into a formula. Unfortunately, it may be necessary to be familiar with Bayes Theorem in a more abstract and general form.

<u>General form of Bayes Theorem</u>

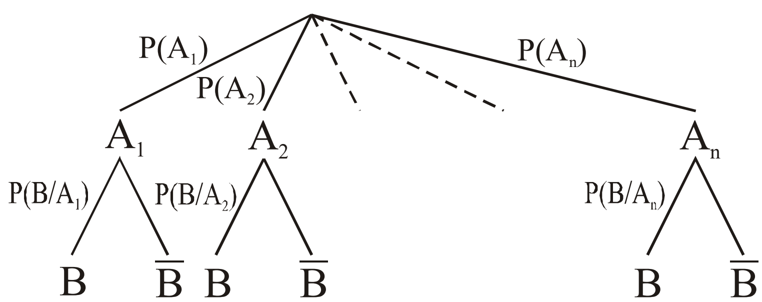
Given that A_1, A_2 …. A_n are mutually exclusive and exhaustive events so that, $P(A_1 \cup A_2 \cup A_n) = 1$ and given that B is another possible event, then for each $i = 1, 2, \ \ldots \ n$

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n)}$$

This is a tedious formula. However the denominator of the fraction (the part on the 'bottom') is just the total probability that event *B* will occur. To explain this let us draw a probability tree.



This shows the events $A_{1,}A_2$ .... A_n followed by the events *B* or not *B* (that is, $\overline{B}$). Since $A_{1,}A_2$ .... A_n are mutually exclusive they form separate branches of the tree; being exhaustive means these are all the branches that there are. We place the probabilities, as usual, against the branches of the tree.



So summing the total probability that B will occur we get the usual

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n)$$

(This represents what one is doing when drawing a probability tree without really thinking about it).

We will now prove the formula in the next example

Example

If $A_{1,}A_2$ ... A_n are mutually exclusive and exhaustive events and B is another arbitrary event, show that

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{\sum_{r=1}^{n} P(A_r)P(B|A_r)}$$

Solution

From the definition of the conditional probability

$$P\left(A_1 \middle| B\right) = \frac{P\left(A_1 \cap B\right)}{P\left(B\right)}$$

The denominator just is

$$P\left(B\right) = \sum_{r=1}^{n} P\left(A_r\right) P\left(B \middle| A_r\right)$$

as can be seen from the tree.

The numerator is also read from the tree as

$$P\left(A_1 \cap B\right) = P\left(A_1\right) \times P\left(B \middle| A_1\right)$$

So the theorem is proven