The Binomial Theorem for Rational Indices

Prerequisites

You should already be familiar with the Binomial theorem which states that the expansion of $(a+b)^n$ is given by

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-r}a^{n-r}b^r + ... + \binom{n}{n}b^n$$

where n is a positive integer and

$$^nC_r \text{ or } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

are the *binomial coefficients.* You should also be familiar with Pascal's triangle as a method for generating these binomial coefficients.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Pascal's triangle

Example (1)

(*a*) Use Pascal's triangle to find the expansion of $(1-3x)^6$.

(*b*) In the binomial expansion of $(a+3x)^7$, the modulus of the coefficient of the term in x^3 is 28 times larger than coefficient of the term in x^3 in the binomial expansion of $(1-3x)^6$. Given that $a \neq 0$, find the value of *a*.

Solution

(*a*) $(1-3x)^6 = \left(1\times1^6\times(-3x)^0\right)+\left(6\times1^5\times(-3x)^1\right)+\left(15\times1^4\times(-3x)^2\right)$
$+\left(20\times1^3\times(-3x)^3\right)+\left(15\times1^3\times(-3x)^4\right)+\left(6\times1^1\times(-3x)^5\right)+\left(1\times1^0\times(-3x)^6\right)$
$=1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$

(*b*) From part (*a*) the modulus of the coefficient of the term in x^3 in the binomial expansion of $(1-3x)^6$ is 540. The coefficient of the term in x^3 in the binomial expansion of $(a+3x)^7$ is

$$\binom{7}{3}\times a^4\times3^3$$

Therefore

$$\binom{7}{3}\times a^4\times3^3=28\times540$$
$$35a^4\times27=15120$$
$$a^4=16$$
$$a=2 \qquad a>0$$

The purpose of this chapter is to extend the range of the Binomial theorem so that the expansion

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+....+\binom{n}{n-r}a^{n-r}b^r+....+\binom{n}{n}b^n$$

shall be valid (1) if n is n is negative and (2) if n is a rational number (that is, a fraction). We will also show how the Binomial theorem can be used to find approximations to irrational numbers such as $\sqrt{25.1}$.

The Binomial theorem when *n* is rational

The Binomial theorem can be extended to cover the expansion of $(1+x)^n$ where n is a rational number (a fraction), provided that $|x|<1$. In that case, the theorem takes the form

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+....$$

Note, this covers both the cases where (1) is n is negative and (2) if n is a rational number (that is, a fraction). The condition $|x|<1$ is necessary in order to make this series convergent – that is, if $|x|<1$, each successive term in x is smaller than the previous one, so the series gets closer and closer to the true value of $(1+x)^n$. For this reason, when the expression is in the form $(a+x)^n$ it must first be placed in the form $a^n\left(1+\frac{x}{a}\right)^n$ before the theorem can be applied; it is valid only when $\left|\frac{x}{a}\right|<1$ or $|x|<a$. A number of examples will clarify the use of the Binomial theorem in this form.

Example (2)

Expand $(1-2x)^{-2}$ as a series of ascending powers of x up to and including the term in x^3. State the set of values of x for which the expansion is valid.

Solution

We use the Binomial theorem in the form $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + ...$

Replacing x in the above by $(-2x)$ and substituting $n=-2$

$$\begin{aligned}(1+(-2x))^{-2} &= 1+(-2)(-2x)+\frac{(-2)(-2-1)(-2x)^2}{2!}+\frac{(-2)(-2-1)(-2-2)(-2x)^3}{3!}+....\\ &= 1+(-2)(-2x)+\frac{(-2)(-3)(-2x)^2}{2!}+\frac{(-2)(-3)(-4)(-2x)^3}{3!}+....\\ &= 1+4x+12x^2+32x^3+....\end{aligned}$$

This expansion is valid for

$$|2x|<1$$

$$|x|<\frac{1}{2}$$

$$-\frac{1}{2}<x<\frac{1}{2}$$

Example (4)

Expand $\frac{1}{\sqrt{1+3x}}$ as a series of ascending powers of x up to and including the term in x^3, expressing the coefficients in their simplest form. State the set of values of x for which the expansion is valid.

Solution

$$\begin{aligned}\frac{1}{\sqrt{1+3x}} &= (1+3x)^{-\frac{1}{2}}\\ &= 1+\frac{\left(-\frac{1}{2}\right)}{1!}(3x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(3x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(3x)^3+\ldots\\ &= 1-\frac{3}{2}x+\frac{27}{8}x^2-\frac{135}{16}x^3+\ldots\end{aligned}$$

This expansion is valid for

$|3x| < 1$

$|x| < \frac{1}{3}$

$-\frac{1}{3} < x < \frac{1}{3}$

Example (4)

Expand $(8-x)^{\frac{1}{3}}$ as a series of ascending powers of x up to and including the term in x^3, expressing the coefficients in their simplest form. State the set of values of x for which the expansion is valid.

Solution

We must first write $(8-x)^{\frac{1}{3}}$ in the form $a^n\left(1+\frac{x}{a}\right)^n$.

$$(8-x)^{\frac{1}{3}} = \left(8\left(1-\frac{x}{8}\right)\right)^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}}\left(1-\frac{x}{8}\right)^{\frac{1}{3}}$$

$$= 2\left(1-\frac{x}{8}\right)^{\frac{1}{3}}$$

$$= 2\left\{1+\left(\frac{1}{3}\right)\left(\frac{x}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3+\ldots\right\}$$

$$= 2\left\{1+\frac{1}{24}x-\frac{1}{576}x^2+\frac{5}{20736}x^3+\ldots\right\}$$

$$= 2+\frac{1}{12}x-\frac{1}{288}x^2+\frac{5}{10368}x^3+\ldots$$

This expansion is valid for

$\left|\frac{x}{8}\right| < 1$

$|x| < 8$

$-8 < x < 8$

The next example illustrates a further extension of this form of the Binomial theorem to a new kind of problem; however, no new theory is involved.

Example (5)

Expand $\dfrac{1+x}{\sqrt{1-2x}}$ as a series of ascending powers of x up to and including the term in x^3, expressing the coefficients in their simplest form. State the set of values of x for which the expansion is valid.

Solution

We must bring the denominator to the top and expand using the Binomial theorem.

$$\frac{1+x}{\sqrt{1-2x}} = (1+x)(1-2x)^{-\frac{1}{2}}$$

$$= (1+x)\left(1+\left(-\frac{1}{2}\right)(-2x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-2x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-2x)^3+\ldots\right)$$

$$= (1+x)\left(1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\ldots\right)$$

$$= \left(1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\ldots\right)+x\left(1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\ldots\right) \qquad (*)$$

$$= 1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\ldots+x+x^2+\frac{3}{2}x^3+\frac{5}{2}x^4+\ldots$$

$$1+2x+\frac{5}{2}x^2+4x^3+\ldots$$

The only new step in this question occurs at the line marked by $(*)$ where we expand the two brackets and subsequently collect terms.

The Binomial Theorem and Approximations

The Binomial Theorem can be used to obtain approximations and to evaluate roots.

Example (6)

(*a*) Expand $(25+x)^{\frac{1}{2}}$ as a series of ascending powers of x up to and including the term in x^2, expressing the coefficients in their simplest form. State the set of values of x for which the expansion is valid.

(*b*) By substituting $x = 0.1$ into the result for part (*a*) obtain an approximation to $\sqrt{25.1}$.

Solution

(*a*) $(25+x)^{\frac{1}{2}} = 5\left(1+\frac{x}{25}\right)^{\frac{1}{2}}$

$$= 5\left\{1+\left(\frac{1}{2}\right)\left(\frac{x}{25}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{25}\right)^2+\ldots\right\}$$

$$= 5\left(1+\frac{1}{50}x-\frac{1}{1250}x^2+\ldots\right)$$

This is valid for

$\left|\frac{x}{25}\right|<1$

$|x|<25$

$-25<x<25$

(*b*) $\sqrt{25.1} = (25+0.1)^{\frac{1}{2}}$

$$= 5\left(1+\frac{1}{50}(0.1)-\frac{1}{5000}(0.1)^2+\ldots\right)$$

$$= 5(1+0.002-0.000002)$$

$$= 5.00999$$

Example (7)

Expand $(1-x)^{-\frac{1}{3}}$ in ascending powers of x up to and including the term in x^2. State the range of values of x for which the expansion is valid. Hence, by writing $x=\frac{1}{8}$, find an approximate value for $\sqrt[3]{7}$ in the form $\frac{a}{b}$ where a and b are integers.

Solution

$$(1-x)^{-\frac{1}{3}} = 1-\frac{1}{3}(-x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}(-x)^2+\ldots$$

$$= 1+\frac{1}{3}x+\frac{2}{9}x^2+\ldots$$

This is valid for

$|x|<1$

$-1<x<1$

Substituting $x=\frac{1}{8}$ into both sides of this

$$\left(1-\frac{1}{8}\right)^{-\frac{1}{3}} \approx 1+\frac{1}{3}\times\frac{1}{8}+\frac{2}{9}\left(\frac{1}{8}\right)^2$$

$$\frac{1}{\sqrt[3]{1-\frac{1}{8}}} \approx 1+\frac{1}{24}+\frac{1}{288}$$

$$\frac{\sqrt[3]{7}}{2} \approx \frac{301}{288}$$

$$\sqrt[3]{7} \approx \frac{301}{144}$$