Calculus and Kinematics

Prerequisites

You should already have studied (1) velocity-time graphs and (2) the equations of uniform acceleration. You should also have a good working knowledge of basic calculus and be able to differentiate and integrate polynomial functions.

Example (1)

A car accelerates from an initial velocity of 11 ms^{-1} to a final velocity of 25 ms^{-1} over a distance of 1000 m. Assuming that the acceleration is constant

(*a*) Find the time taken to do this.

(*b*) Find the acceleration of the car.

(*c*) Draw a velocity-time graph to represent this journey.

Solution

(*a*) The problem may be recast as

Given that $s = 1000$, $u = 11$ and $v = 25$, find t.

Substituting into the equation

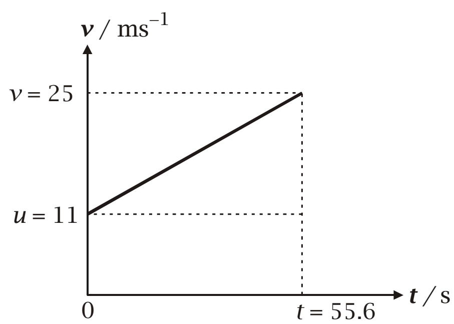
$$s = \frac{1}{2}(v+u)t$$

$$1000 = \frac{1}{2}\times(25+11)t$$

$$t = \frac{1000}{18} = \frac{500}{9} = 55.6\text{ s} \quad (3\text{ s.f.})$$

(*b*) $$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{25-11}{\left(\frac{1000}{18}\right)} = 0.252\text{ ms}^{-2}$$

(*c*)



Example (2)

(*a*) Integrate $f(t) = 0.252$.

(*b*) (*i*) Given $v(0) = 11$ find the exact solution to the equation

$$v(t) = \int f(t)\,dt$$

(*ii*) Find *t* when $v(t) = 25$

(*c*) Integrate the function $v(t)$ found in the solution to part (*b*).

(*d*) (*i*) Given $s(0) = 0$ find the exact solution to the equation

$$s(t) = \int v(t)\,dt$$

(*ii*) Find the value of *s* when $v(t) = 25$.

Solution

(*a*) $f(t) = 0.252$

$\int f(t)\,dt = 0.252t + c \qquad c = \text{constant}$

(*b*) (*i*) From part (*a*)

$v(t) = 0.252t + c$

Substituting $t = 0,\ v = 11$

$c = 11$

$v(t) = 0.252t + 11$

(ii) Substituting $v(t) = 25$

$25 = 0.252t + 11$

$t = \dfrac{25 - 11}{0.252} = 55.6 \text{ s } (3 \text{ s.f.})$

(*c*) $v(t) = 0.252t + 11$

$\int v(t)\,dt = 0.126t^2 + 11t + c \qquad c = \text{constant}$

(*d*) (*i*) From part (*c*)

$s(t) = \int v(t)\,dt = 0.126t^2 + 11t + c$

$s(0) = 0 \Rightarrow c = 0$

$s(t) = 0.126t^2 + 11t$

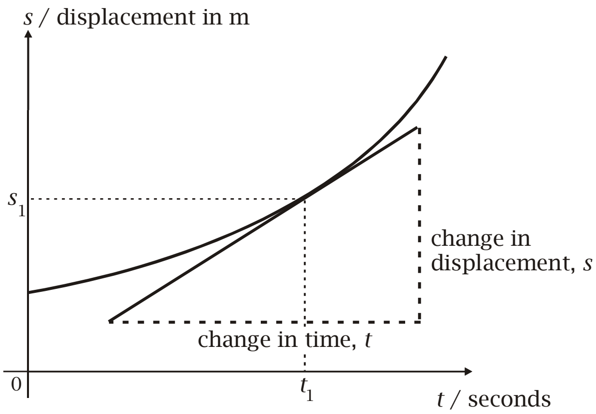
(*ii*) From part (*b*) when $v = 25,\ t = \dfrac{14}{0.252}$

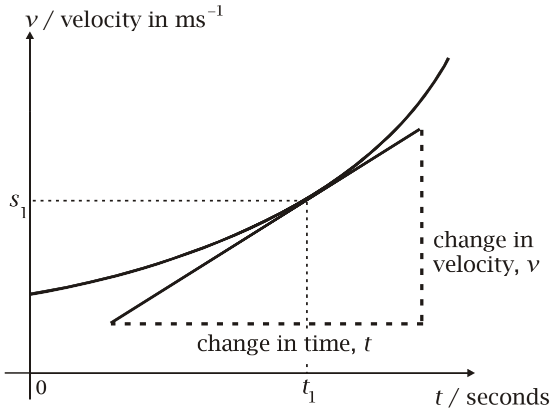
$$s(t) = 0.126\left(\frac{14}{0.252}\right)^2 + 11 \times \frac{14}{0.252} = 1000 \text{ m}$$

The subject of *kinematics* is concerned with the motion of particles. Initially, we study the motion of particles in one-dimension (motion in a straight-line), and subsequently develop a theory for two or three dimensions. Examples (1) and (2) indicate that there is a close relationship between calculus on the one hand and kinematics on the other.

Calculus and kinematics

Velocity is the instantaneous rate of change of displacement with respect to time. Acceleration is the instantaneous rate of change of velocity with respect to time. That is, velocity is the gradient of the tangent to the displacement-time graph for a particle in motion, and acceleration is the gradient of the tangent to the velocity-time graph for a particle in motion.





$$\text{velocity} = \frac{\text{change in displacement}}{\text{change in time}} \qquad \text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

Another way of putting this is to say that velocity is the rate of change of displacement with respect to time, and acceleration is the rate of change of velocity with respect to time. Displacement, velocity and acceleration are functions of time. In calculus the rate of change of a function is found by the process of taking the derivative. Hence

Velocity is the derivative of displacement with respect to time
Acceleration is the derivative of velocity with respect to time

If $s = s(t)$ represents the displacement of a particle (in one dimension) as a function of time, then these relations between displacement, velocity and acceleration are expressed symbolically by

$$v(t) = \frac{ds}{dt}$$
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Example (3)

A particle's displacement is given by

$s(t) = 4.9t^2 + 9t + 3$

Find its velcity and acceleration at $t = 3$. Find also its initial velocity.

Solution

$s(t) = 4.9t^2 + 9t + 3$

$v(t) = \frac{d}{dt}(4.9t^2 + 9t + 3) = 9.8t + 9$

$a(t) = \frac{d}{dt}(9.8t + 9) = 9.8$

When $t = 3$, the velocity is

$v(3) = 9.8 \times 3 + 9 = 38.4 \text{ ms}^{-1}$

When $t = 3$, the acceleration is

$a(3) = 9.8 \text{ ms}^{-2}$

The intial velocity is found by substituting $t = 0$ into the equation for velocity.

$v(0) = 9.8 \times 0 + 9 = 9ms^{-1}$

Integration

Since integration is the inverse process of differentiation, it follows that

Velocity is the integral of acceleration

Displacement is the integral of velocity

When we integrate we must introduce a constant of integration. In the context of mechanics these constants of integration have specific meaning. When integrating acceleration, the constant of integration is the initial velocity of the particle. When integrating velocity, the constant of integration is the initial displacement of the particle. We express this symbolically by

Given $a(t)$, then

$v(t) = \int a(t)\,dt + v_0$

where v_0 is the initial velocity of the particle; and

$s(t) = \int v(t)\,dt + s_0$

where s_0 is the initial displacement of the particle.

Example (4)

A particle moves in a straight line such that its acceleration $a\ \text{ms}^{-2}$ is given by

$a = 10 - 8t \qquad t \geq 0$

At time $t = 0$ the particle is at the point O and its velocity is $6\ \text{ms}^{-1}$.

(a) Find an expression for the velocity of the particle at time t s.

(b) Find an expression for the displacement of the particle from O at time t s.

(c) Determine the time when the particle comes to rest instantaneously and the distance of the particle from O at this time.

(d) Calculate the **speed** of the particle when $t = 2.5$ s, and determine whether or not the **speed** of the particle is increasing or decreasing at this time.

Solution

(a)
$$a = 10 - 8t$$
$$v(t) = \int 10 - 8t\, dt = 10t - 4t^2 + v_0$$
$$v(0) = 6 \Rightarrow v_0 = 6$$
$$v(t) = -4t^2 + 10t + 6$$

(b)
$$s(t) = \int -4t^2 + 10t + 6 = -\frac{4}{3}t^3 + 5t^2 + 6t + s_0$$
$$s(0) = 0 \Rightarrow s_0 = 0$$
$$s(t) = -\frac{4}{3}t^3 + 5t^2 + 6t$$

(c) At instantaneous rest $v(t) = 0$. Therefore
$$-4t^2 + 10t + 6 = 0$$
$$-2t^2 + 5t + 3 = 0$$
$$(3 - t)(2t + 1) = 0$$
$$t = 3 \text{ or } t = -\frac{1}{2}$$
But $t \geq 0$, therefore $t = 3$ s

When $t = 3$, $s(3) = -\frac{4}{3}(3)^3 + 5(3)^2 + 6 \times 3 = 27$ m

(d) When $t = 2.5$
$$v(2.5) = -4(2.5)^2 + 10(2.5) + 6 = 6\ \text{ms}^{-1}$$
$$a(2.5) = 10 - 8(2.5) = -10 < 0$$

The speed is $6\ \text{ms}^{-1}$ and the acceleration, which is negative, is in the opposite direction to the speed, indicating that the particle is decelerating. The particle has not yet reached instantaneous rest and the speed is decreasing.

Example (5)

A particle is thrown vertically upwards from the top of a cliff some 40 m above sea level. If its initial velocity is 5 ms^{-1}, find the maximum height that it rises to, and the time taken to reach the sea. Assume that it can clear the edge of the cliff, as is falls back down, and take the acceleration due to gravity to be 9.8 ms^{-2}.

Solution

$a(t) = -9.8$

The negative sign here is essential because acceleration is a vector. The object is initially thrown upwards but it is accelerating under the effect of gravity at all times. Here we have defined the positive direction to be upwards, hence the acceleration takes a negative sign.

$v(t) = \int -9.8dt = -9.8t + v_0$

Here $v_0 = 5\text{ ms}^{-1}$, hence

$v(t) = -9.8t + 5$

Integrating again gives

$s(t) = \int(-9.8t + 5)dt = -4.9t^2 + 5t + x_0$

The cliff is 40 m above sea-level, so $s_0 = 40$. Thus

$s(t) = -4.9t^2 + 5t + 40$

The particle gains its maximum height when $v(t) = 0$. That is, when

$v(t) = -9.8t + 5 = 0$

$$t = \frac{5}{9.8} = 0.5102...$$

The height it has risen at this time is found by substituting $t = 0.5102...$ into the equation for displacement, $s(t) = -4.905t^2 + 5t + 40$, to get

$$s(0.5102...) = -4.9 \times (0.5102...)^2 + 5 \times 0.5102... + 40 = 41.2742... = 41.2\text{ m } (3\text{ s.f.})$$

The particle reaches sea level when $s(t) = 0$. Substituting into the equation for displacement gives $-4.9t^2 + 5t + 40 = 0$. Solving this by means of the quadratic formula gives

$-4.9t^2 + 5t + 40 = 0$

$$t = \frac{-5 \pm \sqrt{(5)^2 - 4 \times (-4.9) \times 40}}{2 \times (-4.9)}$$

$t_1 = 2.392... \qquad t_2 = -3.413...$

The negative solution does not make sense here. It would correspond to a time before the object was thrown upwards if in fact it had been projected from the bottom of the cliff to begin with. Since it makes no sense in the given physical context we simply discard it. Hence, the solution is

$t = 2.39$ (3 s.f.)

Newton's second law

In many of the applications of calculus to the motion of a particle we are given a situation of a particle moving under gravity, and we take the acceleration to be the acceleration due to freefall $a(t) = g = 9.8\ \text{ms}^{-2}$ or a suitable approximation. However, the use of calculus in kinematics provides us with the flexibility to deal with questions where the acceleration is provided by a resultant force and is not directly given. In this case, we apply Newton's second law in the form

$F = ma$

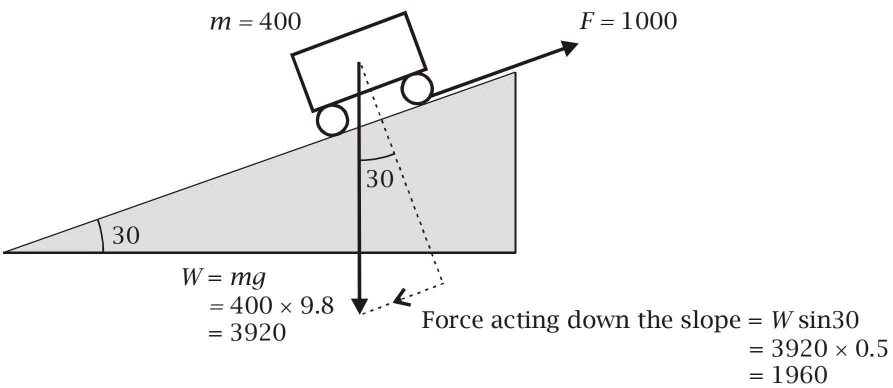
Force = mass $\times$ acceleration

to find the acceleration of the particle. We then use the integral calculus, together with boundary conditions providing information about the initial velocity and displacement, to determine the equations of velocity and displacement for the particle. Rearrangement of Newton's second law gives $a = \frac{F}{m}$, so given the force acting on the particle and its mass, we can find its acceleration.

Example (6)

A runaway truck of mass 400 kg is rolling down a hill inclined at 30° to the horizontal. This motion is resisted by a constant frictional force of 1000 N. Find the car's velocity when it collides with a brick wall 500 m from the point where it initially started to roll.

Solution



As the diagram illustrates, the weight of the truck is

$W = mg = 400 \times 9.8 = 3920$ N

The force acting down the slope is given by

$S = W\sin 30 = 3920 \times 0.5 = 1960$ N

The frictional force is

$F = 1000\,\text{N}$

Hence, the resultant force causing the truck to accelerate down the slope is

$R = S - F = 1960 - 1000 = 960$ N

Applying this figure to Newton's second law

$$a(t) = \frac{R}{m} = \frac{960}{400} = 2.4 \text{ ms}^{-2}$$

Then integrating twice gives

$v(t) = 2.4t + v_0$

$s(t) = 1.2t^2 + v_0 t + s_0$

The truck starts accelerating from rest, so its displacement at $t = 0$ is $s_0 = 0$ and its velocity is also $v_0 = 0$. Hence

$s(t) = 1.2t^2$

is the equation of displacement for the rolling truck.

The truck has 500 m to roll, so substituting $s(t) = 500$ in this equation gives

$1.2\,t^2 = 500$

$$t = \sqrt{\frac{500}{1.2}} = 20.41...s$$

When $t = 20.41...$

$v(20.41...) = 2.4 \times 20.41... = 48.98... = 49.0 \text{ ms}^{-1}$ (3 s.f.)

So it is rolling rather quickly!

The distinction between kinematics and dynamics

The term *kinematics* refers to the study of motion without reference to forces; in *dynamics* motion is studied in the context of forces. Therefore, when we introduced Newton's second law we extended our results from kinematics to dynamics.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