Calculus and Kinematics in Three Dimensions

Prerequisites

You should already have studied (1) basic vector algebra in three dimensions, including being able to use the scalar (dot) product to find the angle between two vectors and (2) calculus and kinematics in one-dimension

Example (1)

Given

$\mathbf{p} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
$\mathbf{q} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(*a*) (*i*) Find **p** + **q** in column form.

(*ii*) Find **p** – **r** in row form.

(*iii*) Find **q** – **r** in **i**, **j** form.

(*b*) Find the acute angle between the vector **p** and the vector **q** – **r**.

Solution

(a) (i) $$\mathbf{p} + \mathbf{q} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

(ii) $\mathbf{p} - \mathbf{r} = (2,-1,1) - (1,2,-1) = (1,-3,2)$

(iii) $\mathbf{q} - \mathbf{r} = (-\mathbf{i} + \mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

(b) Let θ be the angle between **p** and **q** – **r**. Then

$$\cos\theta = \frac{\mathbf{p} \cdot (\mathbf{q} - \mathbf{r})}{|\mathbf{p}| \times |\mathbf{q} - \mathbf{r}|} = \frac{(2,-1,1) \cdot (-2,-1,4)}{|(2,-1,1)| \times |(-2,-1,4)|} = \frac{-4+1+4}{\sqrt{6} \times \sqrt{21}} = \frac{1}{3\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{1}{3\sqrt{14}}\right) = 84.9° \text{ (1 d.p.)}$$

Example (2)

A particle moves in a straight line such that its acceleration a ms^{-2} is given by

$a = 6t - 12 \qquad t \geq 0$

At time $t = 0$ the particle is at the point O and its velocity is 9 ms^{-1}.

(*a*) Find an expression for the velocity of the particle at time t s.

(*b*) Find an expression for the displacement of the particle from O at time t s.

(*c*) Determine the times when the particle comes to rest instantaneously and the distance of the particle from O at these times.

(*d*) Find the acceleration, velocity and displacement of the particle when $t = 2$.

(*e*) Determine the direction in which the particle is travelling and whether the particle is speeding up or slowing down in the following intervals.

(*i*) $0 \leq t \leq 1$
(*ii*) $1 \leq t \leq 2$
(*iii*) $2 \leq t \leq 3$
(*iv*) $t > 3$

Solution

(*a*) $a = 6t - 12 \qquad t \geq 0$

$v(t) = \int 6t - 12\,dt = 3t^2 - 12t + v_0$

$v(0) = 9 \Rightarrow v_0 = 9$

$v(t) = 3t^2 - 12t + 9$

(*b*) $s(t) = \int 3t^2 - 12t + 9 = t^3 - 6t^2 + 9t + s_0$

$s(0) = 0 \Rightarrow s_0 = 0$

$s(t) = t^3 - 6t^2 + 9t$

(*c*) At instantaneous rest $v(t) = 0$. Therefore

$3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$t = 1$ or $t = 3$

So the particle comes to a state of instantaneous rest on two occasions.

When $t = 1$, $s(1) = (1)^3 - 6(1)^2 + 9 \times 1 = 4$ m

When $t = 3$, $s(3) = (3)^3 - 6(3)^2 + 9 \times 3 = 0$ m

(*d*) When $t = 2$

$a(2) = 6 \times 2 - 12 = 0$ ms^{-2}

$v(2) = 3(2)^2 - 12 \times 2 + 9 = -3$ ms^{-1}

$s(2) = (2)^3 - 6(2)^2 + 9 \times 2 = 2$ m (*e*) (*i*) $0 \le t \le 1$

The particle is moving from the origin in the positive direction.
Its velocity is positive, its acceleration is negative, and it is decelerating.
At $t = 1$ it comes to an instantaneous rest at $s = 4$ m from the origin.

(*ii*) $1 \le t \le 2$

The particle is moving back towards the origin.

Both the acceleration and the velocity are negative.

Its speed is increasing from 0 to 3 ms^{-1} when it reaches the $s = 2$ m at $t = 2$ s.

It is accelerating. The acceleration is 0 at $t = 2$ s.

(*iii*) $2 \le t \le 3$

The particle continues to move back towards the origin.

The acceleration is positive and the velocity is negative.

It is slowing down from a speed of 3 ms^{-2} at $t = 2$ s.

At $t = 3$ s it reaches another moment of instantaneous rest at the orign.

(*iv*) $t > 3$

The particle is moving in the positive direction.

Both the acceleration and velocity are positive. It is speeding up.

It continues to accelerate in the positive direction indefinitely.

We will now extend these ideas regarding calculus and kinematics to the three dimensional case.

Motion of a particle in three dimensions

Position

In three dimensions the position of a particle is represented by a *position vector* relative to the origin.

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

where $x = x(t),\ y = y(t),\ z = z(t)$ are the coordinates of the particle, t is time (usually in seconds), and **i**, **j**, **k** are a set of right-handed orthogonal vectors, (usually calibrated in metres).

Example (3)

At time t s, a particle P has position vector **r** m with respect to the origin O given by

$\mathbf{r} = (t-2)\mathbf{i} + (2t-5)\mathbf{j} + (8-t)\mathbf{k}$

Find the distance of P from the origin at $t = 3$ s.

Solution

When $t = 3$ s we have

$$\mathbf{r}(3) = (3-2)\mathbf{i} + (2\times 3-5)\mathbf{j} + (8-3)\mathbf{k} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

The distance from the origin is the modulus (size) of this position vector.

$$\left|\overrightarrow{OP}\right| = |\mathbf{i} + \mathbf{j} + 5\mathbf{k}| = \sqrt{1^2 + 1^2 + 25^2} = \sqrt{27} = 3\sqrt{3} \text{ m}$$

Velocity

Velocity is the derivative of the position vector

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \frac{d}{dt}x\,\mathbf{i} + \frac{d}{dt}y\,\mathbf{j} + \frac{d}{dt}z\,\mathbf{k}$$

To find the velocity of a particle differentiate the individual coordinate functions. The speed of the particle is the modulus (size) of the velocity vector.

Acceleration

Acceleration is the derivative of the velocity vector

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d^2}{dt^2}\mathbf{r} = \frac{d^2}{dt^2}x\,\mathbf{i} + \frac{d^2}{dt^2}y\,\mathbf{j} + \frac{d^2}{dt^2}z\,\mathbf{k}$$

Example (4)

At time t s, a particle P has position vector $\mathbf{r}$ m with respect to the origin O given by

$$\mathbf{r} = \left(t^2 - 1\right)\mathbf{i} + (t+1)\mathbf{j} + \left(t^2 - 2t - 2\right)\mathbf{k}$$

Find the speed and acceleration of P from the origin at $t = 2$ s.

Solution

When $t = 2$ s we have

$$\begin{aligned}\mathbf{v}(t) &= \frac{d}{dt}\mathbf{r} \\ &= \frac{d}{dt}x\,\mathbf{i} + \frac{d}{dt}y\,\mathbf{j} + \frac{d}{dt}z\,\mathbf{k} \\ &= \frac{d}{dt}\left(t^2-1\right)\mathbf{i} + \frac{d}{dt}(t+1)\,\mathbf{j} + \frac{d}{dt}\left(t^2 - 2t - 2\right)\mathbf{k} \\ &= 2t\,\mathbf{i} + \mathbf{j} + 2(t-1)\mathbf{k}\end{aligned}$$

At $t = 2$ s the velocity is

$$\begin{aligned}\mathbf{v}(2) &= (2\times 2)\mathbf{i} + \mathbf{j} + 2(2-1)\mathbf{k} \\ &= 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}\end{aligned}$$

The speed is the modulus (size) of this velocity vector.

$$\text{speed} = |4\mathbf{i} + \mathbf{j} + 2\mathbf{k}| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21} \text{ ms}^{-1}.$$

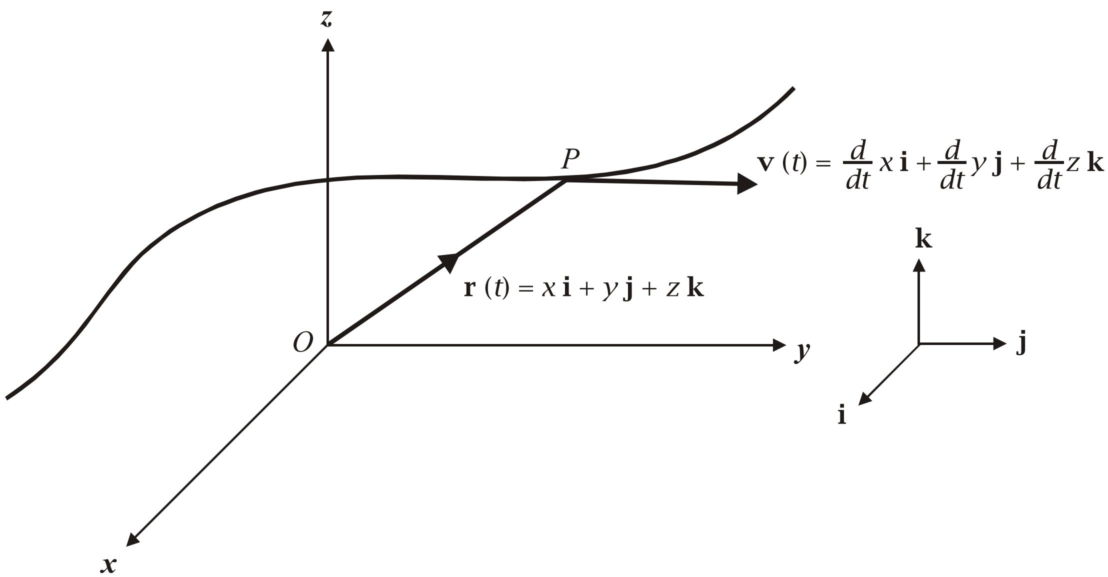
The acceleration is the derivative of the velocity vector.

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}\left(2t\,\mathbf{i} + \mathbf{j} + 2(t-1)\mathbf{k}\right) = 2\,\mathbf{i} + 2\mathbf{k}$$

As the parameter t (time) varies, the position vector

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

traces out a path in three-dimensional space.



Example (5)

At time t s, a particle P has position vector $\mathbf{r}$ m with respect to the origin O given by

$\mathbf{r} = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\mathbf{k}$

(*a*) Show that the path traced out by P is a circle in the plane $z = 2$.

(*b*) Find the velocity of P and find the angle between the velocity vector of P and its position vector.

(*c*) Find the angle made between the acceleration vector of P and the z-axis and the magnitude of the acceleration.

Solution

(*a*) $x(t) = 2\cos t$

$y(t) = 2\sin t$

$z(t) = 2$

Then

$x^2 + y^2 = (2\cos t)^2 + (2\sin t)^2 = 4(\cos^2 t + \sin^2 t) = 4$

The path of P lies on the circle $x^2 + y^2 = 4$ lying in the plane $z = 2$

(*b*) $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\mathbf{k}$

$\mathbf{v}(t) = \dfrac{d}{dt}\mathbf{r}(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$

The angle between **r** and **v** is

$$\begin{aligned}\cos\theta &= \frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{r}||\mathbf{v}|}\\ &= \frac{(2\cos t, 2\sin t, 2)\cdot(-2\sin t, 2\cos t, 0)}{|(2\cos t, 2\sin t, 2)|\times|(-2\sin t, 2\cos t, 0)|}\\ &= \frac{-4\cos t\sin t + 4\sin t\cos t}{|(2\cos t, 2\sin t, 2)|\times|(-2\sin t, 2\cos t, 0)|}\\ &= 0\end{aligned}$$

Hence $\theta = 90°$

So **r** and **v** are perpendicular.

(*c*) $\mathbf{a}(t) = \dfrac{d}{dt}\mathbf{v}(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$

The vector equation of the z-axis is

$\mathbf{u} = s\mathbf{k}$

The angle between **r** and **u** is

$\cos\theta = \dfrac{\mathbf{r}\cdot\mathbf{u}}{|\mathbf{r}||\mathbf{u}|} = \dfrac{(-2\cos t, 2\sin t, 0)\cdot(0, 0, s)}{|\mathbf{r}||\mathbf{u}|} = 0$

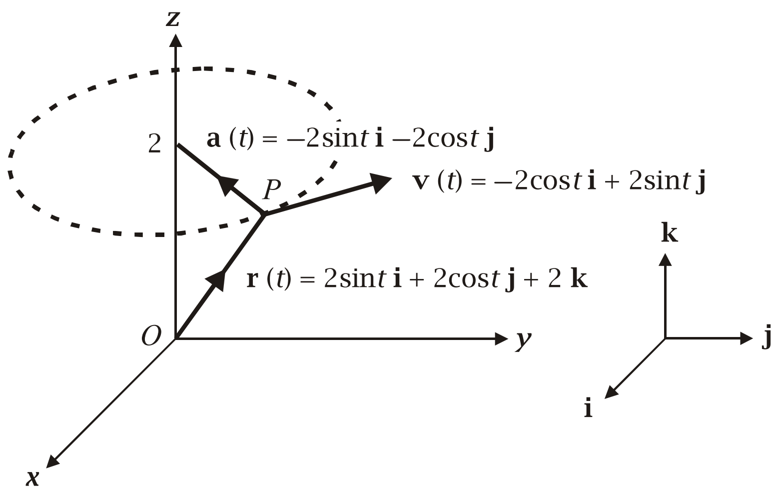
This means that the acceleration is always perpendicular the z-axis.

The magnitude of the acceleration is given by

$|\mathbf{a}| = |-2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}| = \sqrt{2^2 + 2^2 + 0^2} = 2\sqrt{2}\ \text{ms}^{-2}$

The acceleration is constant.

The following diagram illustrates this solution.



Since integration is the reverse process of differentiation, velocity is the integral of acceleration and position is the integral of velocity. When integrating we must introduce constants of integration. As in the one-dimensional case, when acceleration is integrated the constant of integration is the initial velocity of the particle at $t = 0$. When integrating the velocity vector the constant of integration is the initial position of the particle at $t = 0$. These constant of integration are three-dimensional vectors. To find the actual trajectory of the particle one must be given these initial positions or information sufficient to find them. These *initial conditions* are also known as *boundary conditions.*

Example (6)

At time t s, a particle P has acceleration vector $\mathbf{a}$ ms^{-2} with respect to the origin O given by

$\mathbf{a} = 2\mathbf{i}$

(*a*) Given that when $t = 2$ the particle has velocity vector $4\mathbf{i} + 2\mathbf{j}$ find the velocity of the particle at t seconds.

(*b*) Given that when $t = 0$ the particle the particle passes through the point $\left(2, 0, -\sqrt{12}\right)$ find the position vector of the particle at time t seconds.

(*c*) Find the distance of the particle from the origin when it is closest to it.

Solution

(*a*) $\mathbf{a} = 2\mathbf{i}$

$\mathbf{v} = \int \mathbf{a}(t)\,dt = \int 2\mathbf{i}\,dt = (2t + c_1)\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

At $t = 2$ we have

$\mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$

hence

$4\mathbf{i} + 2\mathbf{j} = (4 + c_1)\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

$c_1 = 0 \quad c_2 = 2 \quad c_3 = 0$

The velocity vector of P is therefore given by

$\mathbf{v} = 2t\,\mathbf{i} + 2\,\mathbf{j}$

(*b*) $\mathbf{r} = \int \mathbf{v}(t)\,dt = \int 2t\,\mathbf{i} + 2\,\mathbf{j}\,dt = (t^2 + k_1)\mathbf{i} + (2t + k_2)\mathbf{j} + k_3\mathbf{k}$

At $t = 0$ we have

$\mathbf{r} = 2\mathbf{i} - \sqrt{12}\,\mathbf{k}$

$k_1 = 2 \quad k_2 = 0 \quad k_3 = -\sqrt{12}$

The position vector of P is therefore given by

$\mathbf{r} = (t^2 + 2)\mathbf{i} + 2t\,\mathbf{j} - \sqrt{12}\,\mathbf{k}$

(*c*) The distance of P from the origin is

$$\begin{aligned} OP &= |\mathbf{r}| \\ &= \sqrt{(t^2+2)^2 + (2t)^2 + \left(-\sqrt{12}\right)^2} \\ &= \sqrt{t^4 + 4t^2 + 4 + 4t^2 + 12} \\ &= \sqrt{t^4 + 8t^2 + 16} \\ &= t^2 + 4 \end{aligned}$$

The distance function is therefore

$d = t^2 + 4$

This has a minimum 4 when $t = 0$. So the minimum distance of P from O is 4 m.

A special case is when a particle is travelling along a straight line. The vector $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ traces out the straight line through A and B where $\mathbf{a}$, $\mathbf{b}$ are the position vectors of A, B respectively. This vector $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ has constant speed. An example of a vector representing the motion of a particle along a straight line that does not have constant speed is $\mathbf{r} = \mathbf{a} + t^2(\mathbf{b} - \mathbf{a})$. If a particle passes through a point A with position vector $\mathbf{a}$ at time $t = 0$ seconds and has constant velocity $\mathbf{v}$ then its position at t seconds later is $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.

Example (7)

A particle P travels along a straight line at constant velocity. At $t = 0$ seconds it passes through the point $A(1,1,1)$ and at $t = 1$ seconds it passes through the point $B(2,3,-1)$. Find

(*a*) The position vector of P at time t seconds relative to the origin O.

(*b*) Find the speed of *P*.

(*c*) Another particle *Q* passes through the point $C(1,7,-3)$ at $t=0$ seconds and has constant velocity $\mathbf{v}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$. Show that particles *P* and *Q* collide and find their point of collision.

Solution

(*a*) The position vectors of **a** and **b** are

$\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k} \qquad \mathbf{b}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$

This gives

$$\mathbf{b}-\mathbf{a}=\begin{pmatrix}2\\3\\-1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}-=\begin{pmatrix}1\\2\\-2\end{pmatrix}=\mathbf{i}+2\mathbf{j}-2\mathbf{k}$$

respectively. The position vector of **r** is

$$\begin{aligned}\mathbf{r}&=\mathbf{a}+t(\mathbf{b}-\mathbf{a})\\&=\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}1\\2\\-2\end{pmatrix}\\&=(1+t)\mathbf{i}+(1+2t)\mathbf{j}+(1-2t)\mathbf{k}\end{aligned}$$

(*b*) The speed is the modulus of the velocity vector

$$|\mathbf{v}|=|\mathbf{b}-\mathbf{a}|=\sqrt{1^2+2^2+2^2}=\sqrt{9}=3\text{ ms}^{-1}$$

(*c*) The position vector of *Q* is given by

$$\begin{aligned}\mathbf{r}'&=\mathbf{c}+t\mathbf{v}\\&=\begin{pmatrix}1\\7\\-3\end{pmatrix}+s\begin{pmatrix}2\\1\\-2\end{pmatrix}\\&=(1+2s)\mathbf{i}+(7+s)\mathbf{j}+(-3-2s)\mathbf{k}\end{aligned}$$

If the two particles collide then for some values of t and s they will meet. That means we can equate the position vectors of *P* and *Q* respectively.

$$\mathbf{r}(t)=\mathbf{r}'(s)$$

If in fact they do not collide then it will be impossible to consistently solve this equation. Any attempt to do so will lead to a contradiction. If they do collide there will be a consistent solution. Hence

$\mathbf{r}(t) = \mathbf{r}'(s)$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}1\\7\\-3\end{pmatrix} + s\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$t + 1 = 1 + 2s \qquad (1)$

$1 + 2t = 7 + s \qquad (2)$

$1 - 2t = -3 - 2s \qquad (3)$

Solving (1) and (2) simultaneously we obtain

$t = 4$

$s = 2$

This solution is consistent with (3). Here to state that the solution is consistent means that if we solve, for example, equations (1) and (3) simultaneously we obtain the same solution. That is, whatever pair of equations we solve, the same solution is obtained. This is not the case when the solution is inconsistent. So the particles do collide. Their point of collision is found by substituting either $t = 4$ or $s = 2$ into the appropriate equation for the position of P or Q.

$$\mathbf{r}(4) = \begin{pmatrix}1\\1\\1\end{pmatrix} + 4\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}5\\9\\-7\end{pmatrix} = 5\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$$