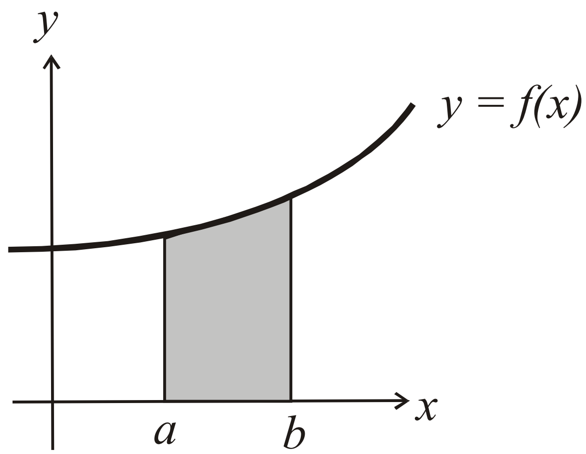
Centre of Mass of a Uniform Lamina

Our aim is to find the centre of mass of a uniform lamina under the curve $y = f(x)$ between the limits $x = a$ and $y = b$. The following diagram illustrates this problem.



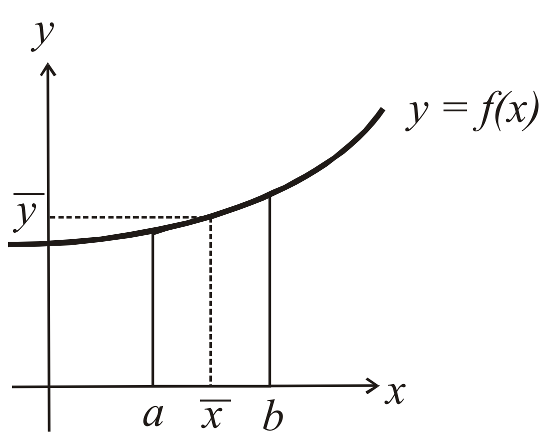
We wish to find the centre of mass of the shaded region.

A lamina is a thin sheet of material. The sheet is sufficiently thin for its thickness to be ignored, and in the formulae and discussion that follows we treat the lamina as having no thickness whatsoever.

Let

$C = (\bar{x}, \bar{y})$

be the position of the centre of mass. The centre of mass is defined to be that point in an object where the turning effect of forces is zero.



Hence, we use the conservation of moments (torque) to find $\bar{x}$ and $\bar{y}$. However, before we do so let us first state and illustrate the main result of this section.

A lamina is bounded by the curve *y = f(x)*, and the lines *x = a, x = b, y = 0.* Then its centre of mass

$$C = (\bar{x}, \bar{y})$$

is given by:-

$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}$$

$$\bar{y} = \frac{\int_a^b \frac{y^2}{2}\,dx}{\int_a^b y\,dx}$$

<u>Example (1)</u>

A uniform lamina is bounded by the *x*-axis, the *y*-axis, the line *x = 2* and the curve $y = 4x^3$.

Find its centre of mass.

$$\int_0^2 y\,dx = \int_0^2 4x^3\,dx = \left[x^4\right]_0^2 = 16$$

$$\int_0^2 xy\,dx = \int_0^2 x4x^3\,dx = \int_0^2 4x^4\,dx = \left[\frac{4x^5}{5}\right]_0^2 = \frac{128}{5}$$

$$\int_0^2 \frac{y^2}{2}\,dx = \frac{1}{2}\int_0^2 \left(4x^3\right)^2\,dx$$

$$= \frac{1}{2}\int_0^2 16x^6\,dx$$

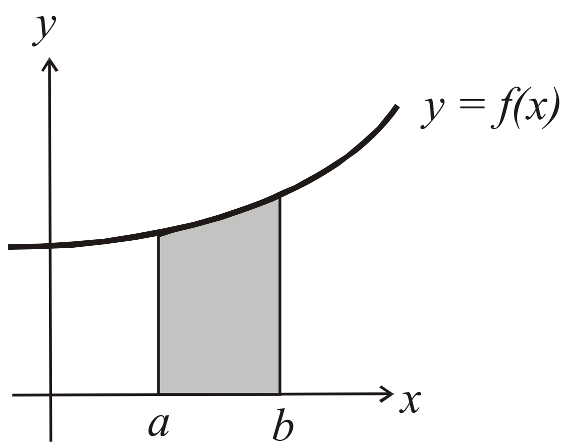
$$= \frac{1}{2}\left[\frac{16x^7}{7}\right]_0^2 = \frac{8\times 128}{7} = \frac{1024}{7}$$

$$\therefore \bar{x} = \frac{\int_0^2 xy\,dx}{\int_0^2 y\,dx} = \frac{128/5}{16} = \frac{8}{5}$$

$$\therefore \bar{y} = \frac{\int_0^2 \frac{1}{2} y^2\,dx}{\int_0^2 y\,dx} = \frac{1024/7}{16} = \frac{64}{7}$$

The proof of the formula is as follows.

To find the centre of mass of a uniform lamina under the curve *y = f(x)*, as shown below.



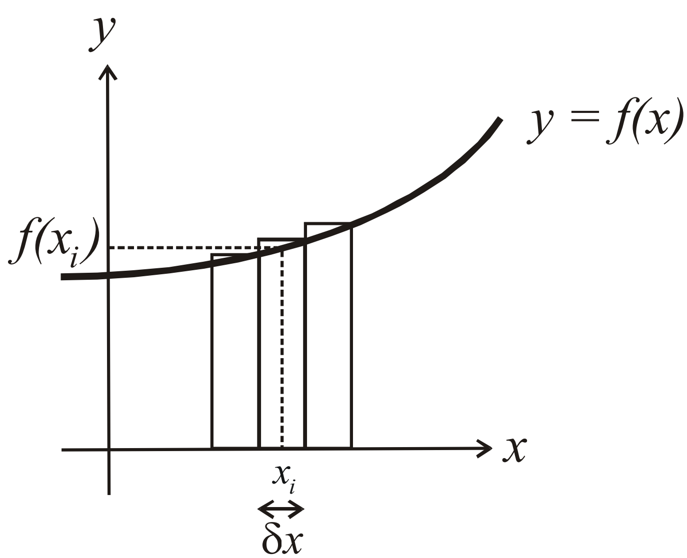
Let

$$\left(\bar{x}, \bar{y}\right)$$

be the centre of mass and ρ the density of the lamina per unit area. The area of the lamina is

$$A = \int_a^b f\left(x\right)dx = \int_a^b y\,dx$$

We divide the lamina into *n* strips of width δx and height *y*.



The area of the *i*th strip is

$$y_i \delta x = f\left(x_i\right)\delta x$$

Since here, mass = density x surface area, the mass of the *i*th strip is:
$M_i = \rho y_i \delta x$

The mass of the whole lamina is

$$M = \rho A = \rho\int_a^b y.dx$$

By conservation of moments, and by taking moments about the *y*-axis.

Sum of the moments of each strip about its centre of mass = the moment of the whole lamina about $\overline{x}$

$$\sum_{i=0}^{n} M_i X_i = M\overline{x}$$

$$\therefore \sum_{i=0}^{n} \rho y_i \partial x \times x_i = \left(\rho\int_a^b ydx\right)\overline{x}$$

Now

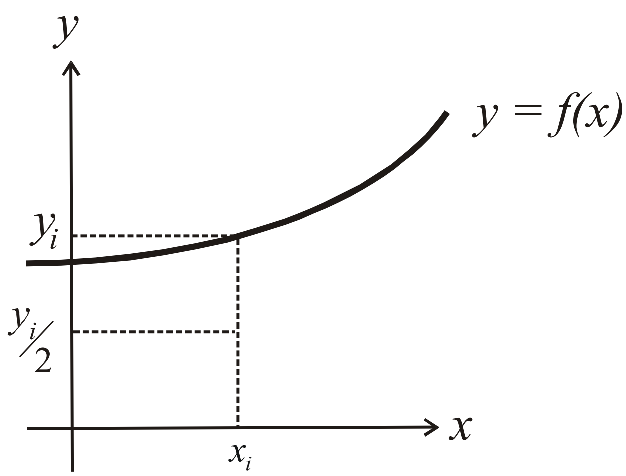
$$\sum_{i=0}^{n} \rho y_i \delta x \times x_i = \sum_{x=a}^{b} \rho y_i \delta x \times x_i$$

Cancelling through by ρ, and taking the limit $dx \to 0$, we obtain:

$$\int_a^b xy\,dx = \bar{x}\int_a^b y\,dx$$

$$\therefore \bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}$$

To find the position of *y*-bar, observe that each strip has *y*-coordinate of centre of mass given by $\frac{y_i}{2}$



Taking moments about the *x*-axis.

$$\sum_{x=a}^{b} \rho y_i \delta x \times \frac{y_i}{2} = \left(\rho \int_a^b y\,dx\right)\bar{y}$$

$$\therefore \int_a^b \frac{y^2}{2}\,dx = \left(\int_a^b y\,dx\right)\bar{y}$$

$$\bar{y} = \frac{\int_a^b \frac{y^2}{2}\,dx}{\int_a^b y.dx}$$