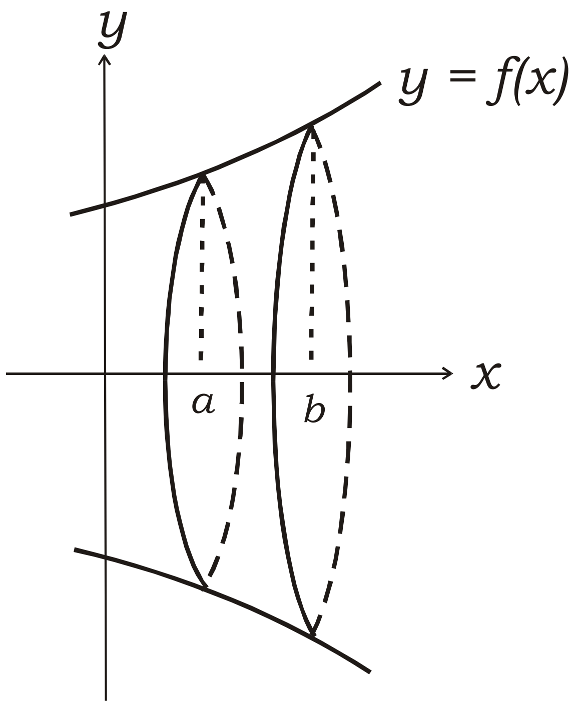
Centre of Mass of a Uniform Solid of Revolution

Our aim is to find the centre of mass of a uniform solid of revolution under the curve $y = f(x)$ between the limits $x = a$ and $x = b$. The following diagram illustrates this problem.



Let

$(\overline{x}, \overline{y})$

be the centre of mass. By symmetry – since the solid is formed by revolution about the x-axis,

$\overline{y} = 0$

The result for $\overline{x}$ is:

$$\overline{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}$$

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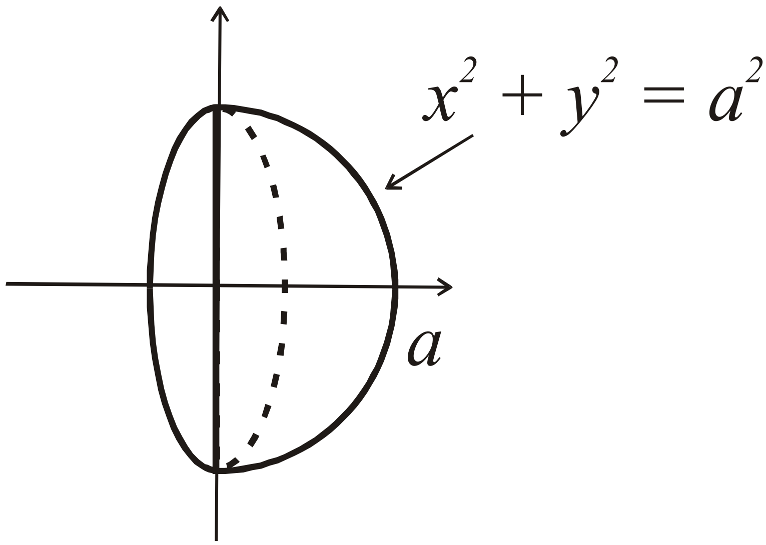
Before we prove this result we illustrate its application.

Example (1)

Prove that the centre of mass of a solid hemisphere of radius a is $3a/8$ from the plane face.



The x-coordinate of the centre of mass is

$$\bar{x} = \frac{\int_0^a xy^2 dx}{\int_0^a y^2 dx}$$

Now $\int_0^a y^2 dx = \int_0^a \left(a^2 - x^2\right) dx$

$$= \left[a^2 x - \frac{x^3}{3}\right]_0^a = a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$$

And $\int_0^a xy^2 dx = \int_0^a x\left(a^2 - x^2\right) dx$

$$= \int_0^a a^2 x - x^3 .dx$$

$$= \left[\frac{a^2 x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{2} - \frac{a^4}{4} = \frac{a^4}{4}$$

$$\therefore \bar{x} = \frac{a^4/4}{2a^3/3} = \frac{3a}{8}$$ Shown

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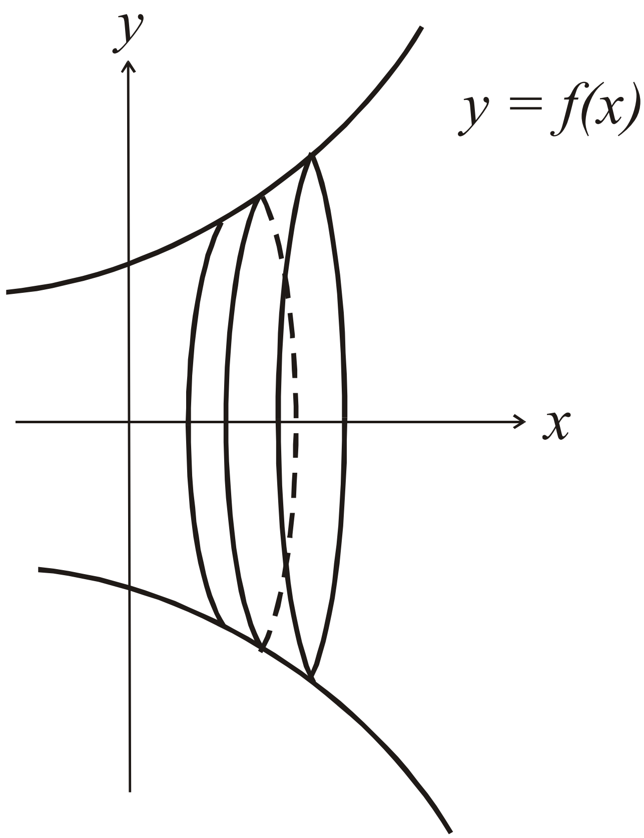
We will now prove the result.



To find the centre of mass of a uniform solid of revolution under the curve $y = f(x)$ between the limits $x = a$ and $y = b$.

Let

$(\bar{x}, \bar{y})$

be the centre of mass and ρ the density of the solid per unit volume. The volume of the solid of revolution is:

$$V = \pi \int_a^b y^2 dx$$

We divide the revolution into n strips of width δx. The height of each strip corresponding to ordinate x_i is y_i.

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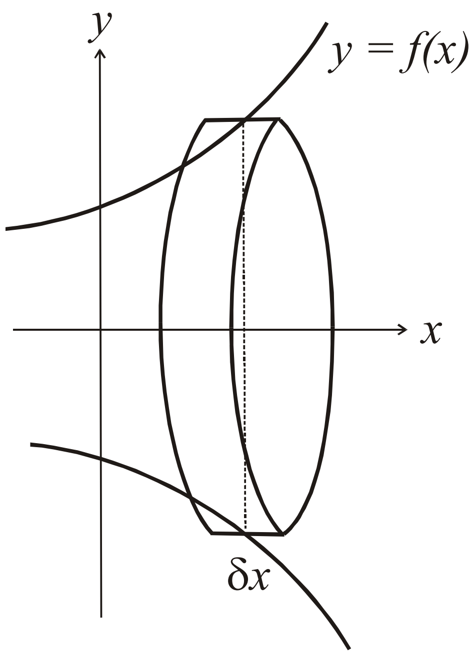

The volume of each strip is

$$\pi\left(y_i\right)^2 dx$$

Since mass = density x volume.

The mass of the *i*th strip is

$$M_i = \rho\pi\left(y_i\right)^2 dx$$

The mass of the whole solid is

$$\rho V = \rho\pi\int_a^b y^2 dx$$

Taking moments about the *y*-axis.

Sum of moments of each section about its centre of mass	=	The moment of the whole solid of revolution about $\overline{x}$

$$\therefore \sum_{i=0}^{n} M_i X_i = M.\overline{x}$$

also $\sum_{i=0}^{n} M_i X_i = \sum_{x=a}^{b} M_i X_i$

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Therefore,

$$\sum_{x=a}^{b} \rho\pi\left(y_i\right)^2 \delta x.X_i = \left(\rho\pi\int_a^b y^2 dx\right)\overline{x}$$

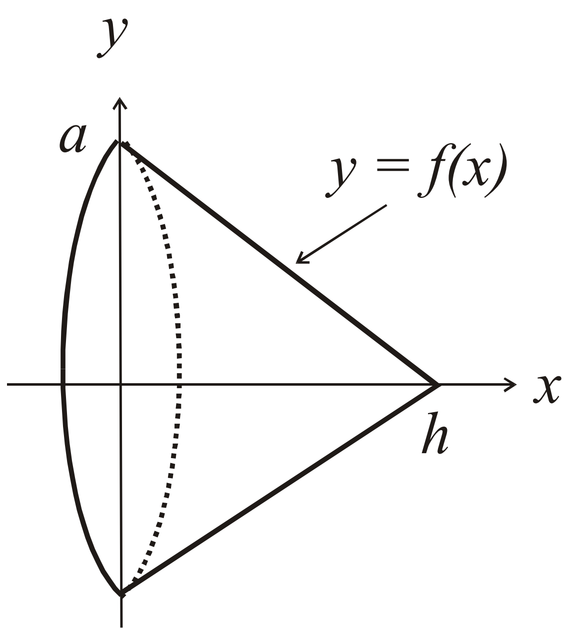
$$\therefore \int_a^b y^2 x dx = \left(\int_a^b y^2 dx\right)\overline{x}$$

$$\therefore \overline{x} = \frac{\int_a^b y^2 .x.dx}{\int_a^b y^2 .dx}$$

We conclude with a further example of applications of this result.

<u>Example (2)</u>

Prove that the centre of mass of a solid cone of height h is $h/4$ from its base.



A solid cone is formed by revolving the straight line $y = f(x)$ as shown. Let the base have area a. To find the equation of $y = f(x)$ note that it is a straight line with equation $y = mx + c$. The gradient is $-a/h$ and the intercept is a. Hence it is:

$$y = -\frac{a}{h}x + c$$

Then

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$$\overline{x} = \frac{\int_0^h y^2 x dx}{\int_0^h y^2 dx}$$

Now $\int_0^h y^2 dx = \int_0^h \left(-\frac{a}{h}x + a \right)^2 dx$

$$= \int_0^h \left(\frac{a^2}{h^2}x^2 - \frac{2a^2}{h}x + a^2 \right) dx$$

$$= \left[\frac{a^2x^3}{3h^2} - \frac{a^2x^2}{h} + a^2 h \right]_0^h$$

$$= \frac{a^2h^3}{3h^2} - \frac{a^2h^2}{h} + a^2 h$$

$$= a^2 \left(\frac{h}{3} - h + h \right)$$

$$= \frac{a^2 h}{3}$$

And

$$\int_0^h y^2 x dx = \int_0^h \left(-\frac{a}{h}x + a \right)^2 x dx$$

$$= \int_0^h \left(\frac{a^2x^3}{h^2} - \frac{2a^2x^2}{h} + a^2 x \right) dx$$

$$= \left[\frac{a^2h^4}{4h^2} - \frac{2a^2x^3}{3h} + \frac{a^2x^2}{2} \right]_0^h$$

$$= a^2 \left(\frac{h^6}{4h^4} - \frac{2h^5}{3h} + \frac{h^2}{2} \right)$$

$$= a^2 \left(\frac{h^2}{4} - \frac{2h^2}{3} + \frac{h^2}{2} \right)$$

$$= \frac{a^2h^2}{12}$$

$\therefore \overline{x} = \dfrac{a^2h^2/12}{a^2h/3} = \dfrac{h}{3}$ Shown.

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