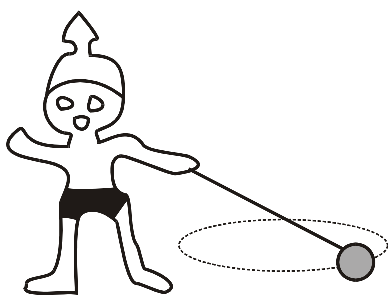
Centripetal Force and Motion in a Horizontal Circle

Prerequisites

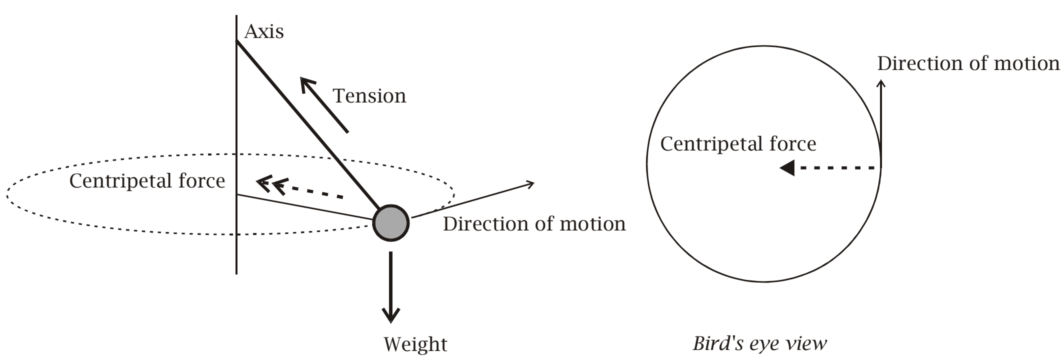
You should be familiar with Newton's second law and with resolving forces, including problems concerning the static equilibrium of an object resting on an inclined surface. You should also be familiar with measuring angles in radians, and finding the arc length of a segment of a circle.

Example (1)

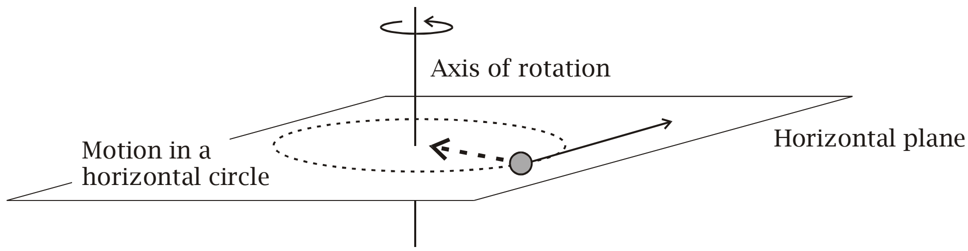


A small boy played an amusing game of whirling a ball on a string in the air. Explain in terms of forces why the ball continues moving in a horizontal circle. Ignore air resistance.

Solution



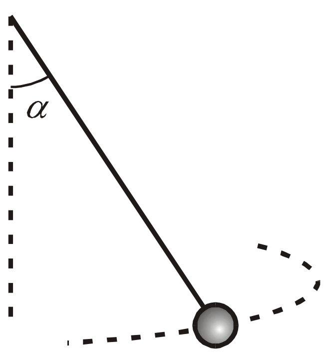
Once the ball is moving the it would naturally want to move in a straight line, indicated in the diagram by the direction of motion. What keeps it moving in a circle is the tension in the string. This tension does two things. Firstly, it holds the ball up. The ball has a weight and would fall but for the tension in the string. Secondly, the tension in the string supplies a *centripetal force* that pulls the ball towards the centre of the circle. Since we are ignoring air resistance there are only two actual forces acting on the ball; these are the tension in the string and the weight of the ball. The centripetal force, which acts in a horizontal plane directed towards the axis of rotation, is the resultant of these two forces. The ball is travelling in a horizontal circle that lies in a plane perpendicular to the axis of rotation.



The *centripetal force* is also called the *radial force*.

Example (2)

A sphere of mass m kg is moving in a horizontal circle. It is held up by a cord inclined at an angle α to the vertical axis of rotation, as shown in the diagram.



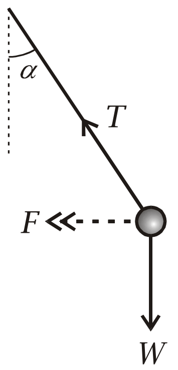
Find in terms of m the magnitude of the tension in the cord and the resultant centripetal force.

Solution

Let the weight of the sphere be W.

Let the tension in the cord be T.

Let the resultant centripetal force be F.



Resolving vertically

$(\uparrow)\qquad T\cos\alpha = W = mg$

$$T = \frac{mg}{\cos\alpha}$$

Resolving horizontally

$(\rightarrow)\qquad F = T\sin\alpha$

$$= \frac{mg}{\cos\alpha} \times \sin\alpha$$
$$= mg\tan\alpha$$

These equations are of general application.

Example (3)

A cyclist in a racing event is travelling at a constant speed in a horizontal circle on a track banked at an angle of 20° to the horizontal. There is no tendency to slip at this speed. The total mass of the rider and the cycle is 100 kg. Modelling the cycle and the rider as a particle

(*a*) calculate the normal reaction of the track on the cycle,

(*b*) calculate the centripetal force acting on the rider and his cycle that keeps them moving in a horizontal circle.

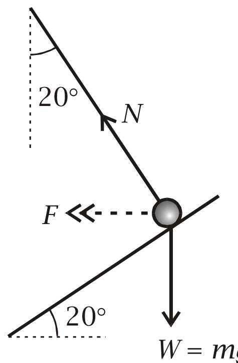
Solution

In this question the object (here the cyclist and his bike) are not held up by a cord, but the force that balances the weight and provides the resultant centripetal force is the normal reaction of the surface of the track.

Let the weight of the sphere be *W*.

Let the normal reaction be *N*.

Let the resultant centripetal force be *F*.



$W = mg = 100 \times 9.8 = 98$ N

(*a*) Resolving vertically

$(\uparrow)$ $N\cos 20° = W = 98$

$$N = \frac{98}{\cos 20°} = 104.28... = 104 \text{ N} \ (3 \text{ s.f.})$$

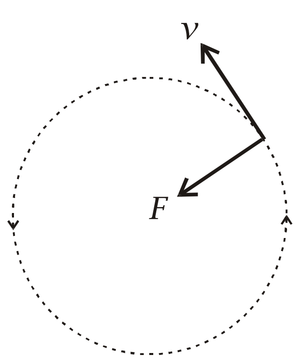
(*b*) Resolving horizontally

$(\rightarrow)$ $F = N\sin 20°$

$$= \frac{98}{\cos 20°} \times \sin 20°$$
$$= 98 \times \tan 20°$$
$$= 35.669...$$
$$= 35.7 \text{ N} \ (3 \text{ s.f.})$$

Motion in a horizontal circle

What we have seen is that when a particle is moving in a circle with constant speed, v ms^{-1}, a force is required to keep it in this circle.



This force must be directed towards the centre of the circle, and is consequently called a centripetal or radial force.

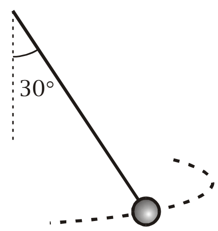
The magnitude of this centripetal force is

$$F = \frac{mv^2}{r}$$

where m kg is the mass of the particle and r m is the radius of the horizontal circle of motion. We shall prove this below.

Example (4)

A sphere of mass 2.5 kg is moving in a horizontal circle at a constant speed of $2.1\ \text{ms}^{-1}$. It is held up by a cord inclined at an angle 30° to the vertical axis of rotation, as shown in the diagram.

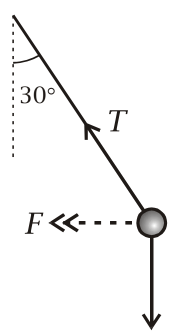


Find

(*a*) the magnitude of the tension in the cord,

(*b*) the radius of the circle.

Solution



$W = mg = 2.5 \times 9.8 = 24.5$ N

(*a*) Resolving vertically

$(\uparrow)\qquad T\cos 30° = W = 24.5$

$$T = \frac{24.5}{\cos 30°} = 28.29... = 28.3 \text{ N } (3 \text{ s.f.})$$

(*b*) Resolving horizontally

$$(\rightarrow)\qquad F = T\sin 20° = \frac{24.5}{\cos 30°} \times \sin 30° = 24.5 \times \tan 30° = 14.145...$$

Then

$$F = \frac{mv^2}{r}$$

$$14.145... = \frac{2.5 \times (2.1)^2}{r}$$

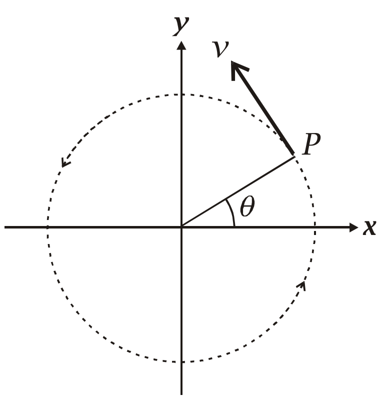
$$r = \frac{2.5 \times (2.1)^2}{14.145...} = 0.7794... = 0.779 \text{ m } (3 \text{ s.f.})$$

Remark

If the speed of the particle is not constant the resultant force will then be either pulling the particle in towards the centre of the horizontal circle in a spiral or the particle will be spiralling outwards. Yet even when the particle's speed is constant it is accelerating. This is because there is a centripetal force that is not zero. This force causes the particle to constantly change its direction. By Newton's first law if the centripetal force did not exist, the particle would otherwise travel in a straight line at constant speed, but the centripetal force causes the motion of the particle to be bent into a circle.

Angular velocity

Let a particle *P* be moving at a constant speed v ms^{-1} in a horizontal circle. Let the origin of the motion be placed at the centre of the circle and let us also establish a set of coordinates for the horizontal plane in which the motion takes place.



Let the angle made by the particle and the *x*-axis be θ. As *P* moves around the circle the angle θ increases. Suppose at time $t = 0$ s the angle is also $\theta = 0$ so that we fix the direction of the *x*-axis by this means. Then clearly as *t* increases the angle θ increases. The rate of increase of the angle θ is also clearly related in some way to the constant velocity of the particle *P*, which is v ms^{-1}.

Angular velocity

We call the angle swept out per unit of time *angular velocity* and usually denote this by ω. (This is a Greet letter pronounced "omega".) We have the following.

$$\text{angular velocity} = \frac{\text{change in angle}}{\text{change in time}} \qquad \omega = \frac{\theta}{t}$$

Angles can be measured in either degrees $\left(\text{symbol }^\circ\right)$ or radians $\left(\text{symbol, rad or }^{c}\right)$. Therefore, depending on how angles are measured the units of angular velocity are either degrees per second $\left(^\circ\text{s}^{-1}\right)$ or radians per second $\left(\text{rad s}^{-1}\right)$. However, it is usual for angular velocity to be measured in radians per second, and this is assumed in questions unless otherwise stated.

We remarked above that there must be a relationship between angular velocity (ω) of a particle in horizontal motion and its speed (v). We state here that this relationship is given by the equation $v = r\omega$ where angular velocity (ω) is measured in radians per second. If we use degrees per second we must convert radians to degrees using the usual equivalence $\pi \text{ rad} \equiv 180^\circ$. We will prove the formula $v = r\omega$ in an appendix below.

Example (5)

A particle *P* is travelling in a horizontal circle of radius 5 m at a speed of 12 ms^{-1}. Find the angular velocity of particle in (*a*) radians per second and (*b*) degrees per second.

Solution

$v = r\omega$

$v = 12 \quad r = 5$

$12 = 5\omega$

$\omega = \frac{12}{5} = 2.4 \text{ rad s}^{-1}$

In degrees per second

$\omega = 2.4 \times \frac{180}{\pi} = 137.509... = 137.5^\circ\text{s}^{-1}$

Example (6)

The minute hand of the *Big Ben* clock is 3 m long. Find the speed of the tip of this minute hand.

Solution

The minute hand makes one revolution in 60 minutes; that is it moves through 2π radians in 60×60 seconds. So its angular velocity is

$$\omega = \frac{2\pi}{60 \times 60} \text{ rad s}^{-1}$$

Its velocity is $v = r\omega = 3 \times \frac{2\pi}{60 \times 60} = 0.00523\ldots \text{ ms}^{-1} = 5.23 \text{ mm s}^{-1}$

Centripetal acceleration

Let a particle P be moving at a constant speed v ms^{-1} in a horizontal circle. Then P is subject to a centripetal force. The centripetal force causes the particle to accelerate towards the centre of the circle of motion. This *centripetal acceleration* has the effect of bending the motion of the particle constantly into a circle. If the centripetal force is increased the particle will spiral in towards the centre, if it is decreased the particle will spiral outwards. Newton's second law is $F = ma$. We have stated that when F is the centripetal force we have $F = \frac{mv^2}{r}$. Hence

$$\frac{mv^2}{r} = ma$$
$$a = \frac{v^2}{r}$$

Since the relationship between speed and angular velocity is $v = r\omega$ this also gives

$$a = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

Remark

The above only serves to illustrate the relationship between centripetal force, acceleration and speed and are **not** proofs. Strictly we should prove both the following formula

(1) $v = r\omega$

(2) $a = r\omega^2$

where $v =$ speed, $r =$ radius, $\omega =$ angular velocity, $a =$ centripetal acceleration

From these the formulae $F = \frac{mv^2}{r} = mr\omega^2$ follow by Newton's second law.

You may take these relationships on trust, or follow the proofs given in the appendix below.

Summary

Centripetal force $F = \frac{mv^2}{r} = mr\omega^2$ Centripetal acceleration $a = \frac{v^2}{r} = r\omega^2$

We remind you that in these formulae angles are measured in radians.

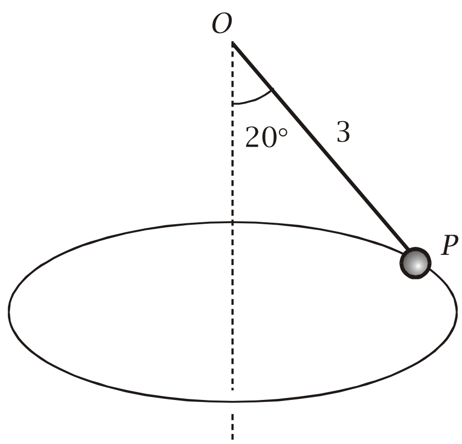
Example (7)

A car is travelling at uniform speed v ms^{-1} on a horizontal track that is a circle of radius 225 m. Given that the centripetal acceleration is 0.25 ms^{-1}, find *v*.

Solution

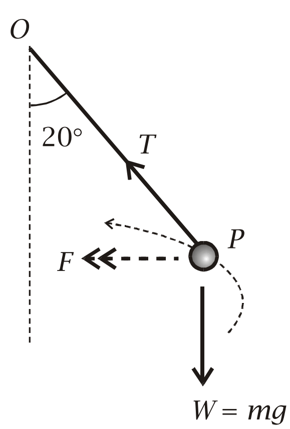
$$a = \frac{v^2}{r} \quad \Rightarrow \qquad v = \sqrt{ar} = \sqrt{0.25 \times 225} = 7.5 \text{ ms}^{-1}$$

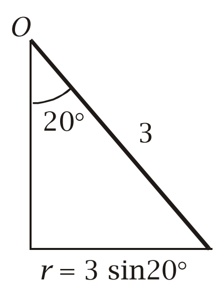
Example (8)



A particle *P* is attached to one end of a light inextensible string of length 3 m. The other end of the string is attached to a fixed point *O*. The particle is moving in a horizontal circle with constant speed v ms^{-1}. The string is taut and makes an angle of 20° to the vertical. Find *v*.

Solution





The centripetal force is being provided by the horizontal component of the tension in the string *T*. The vertical component is in equilibrium with the weight.

$(\uparrow)$	$T\cos 20 = mg$	(1)
$(\rightarrow)$	$T\sin 20 = \dfrac{mv^2}{r}$	(2)
from (1)	$T = \dfrac{mg}{\cos 20}$	

Substituting in (2)

$$\frac{\cancel{m}g}{\cos 20}\times \sin 20 = \frac{\cancel{m}v^2}{r}$$

Therefore

$$v^2 = \tan 20 \times g \times r$$

Here $r = 3\sin 20$

$$v = \sqrt{\tan 20 \times 9.8 \times 3\sin 20} = 1.9130... = 1.91 \text{ ms}^{-1} \text{ (3 s.f.)}$$

Appendix - proofs

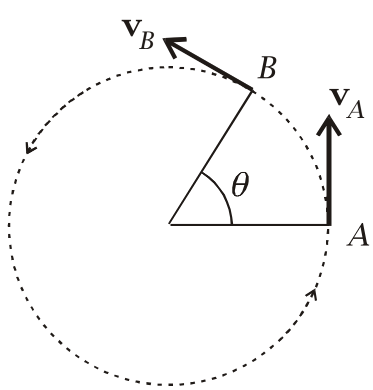
This section is optional. Students are advised learn the proofs wherever possible as these develop insight and help recall.

To prove

(1) $v = r\omega$ (2) $a = r\omega^2$

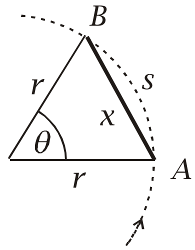
where v = speed, r = radius, ω = angular velocity, a = centripetal acceleration

Proof



Consider a particle moving at constant speed v ms^{-1} in a horizontal circle. Initially it is at point A. After a short interval, t seconds, it reaches B, in which time it has swept out an angle of θ.

In t seconds the displacement between A and B is shown in the following diagram by the straight line joining A to B.



This displacement is approximately equal to the arc length. That is

$x \approx s = r\theta$

In the limit as $t \to 0$ $(t$ gets smaller and smaller$)$ then this becomes exact. Hence

$$\frac{dx}{dt} = \frac{ds}{dt} = \frac{d}{dt}r\theta$$

$$v = r\frac{d\theta}{dt}$$

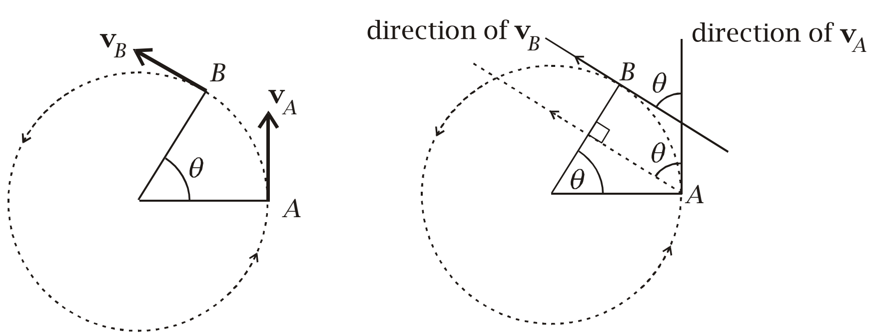
But $\omega = \dfrac{d\theta}{dt}$ is the rate of change of angle per unit of time; that is, angular velocity. Hence

$v = r\omega$

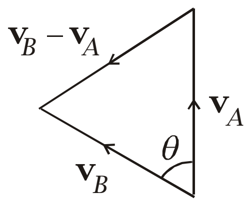
This proves the first formula.

To prove the second formula, let the velocity of the particle P at A be $\mathbf{v}_A$ and at B be $\mathbf{v}_B$. The speed is constant, so $|\mathbf{v}_A| = |\mathbf{v}_B| = v$.

The following diagram shows that the angle between the velocity vectors V_A and V_B is also θ.



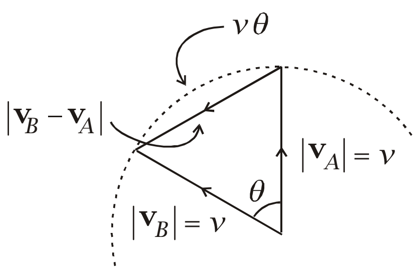
The change in velocity is found by the triangle law for vectors as $\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$.



Over time *t* the average acceleration is

$$a = \frac{v}{t} = \frac{|\mathbf{v}_B - \mathbf{v}_A|}{t}$$

and the size of the instantaneous acceleration is $a = \frac{d}{dt}|\mathbf{v}_B - \mathbf{v}_A|$. Now consider the following diagram



The magnitude of the vector $\mathbf{v}_B - \mathbf{v}_A$ is shown by the straight line joining the tips of $\mathbf{v}_A$ to $\mathbf{v}_B$. The arc length joining the tips of V_A to V_B is $v\theta$. (This follows from the arc length formula $s = r\theta$ where for *s* we substitute the distance along the arc from V_A to V_B and for *r* we substitute the speed *v*.) So we have

$$|\mathbf{v}_B - \mathbf{v}_A| \approx v\theta$$

In the limit as $t \to 0$ then $|\mathbf{v}_B - \mathbf{v}_A| \to v\theta$. That is

$$a = \frac{d}{dt}|\mathbf{v}_B - \mathbf{v}_A| = \frac{d}{dt}v\theta = v\frac{d\theta}{dt} = v\omega$$

From the first relationship $v = r\omega$. Hence

$$a = r\omega^2$$

or

$$a = \frac{v^2}{r}$$

From this the relationship for centripetal force

$$F = ma = m\frac{v^2}{r} = mr\omega^2$$

follows by Newton's second law.