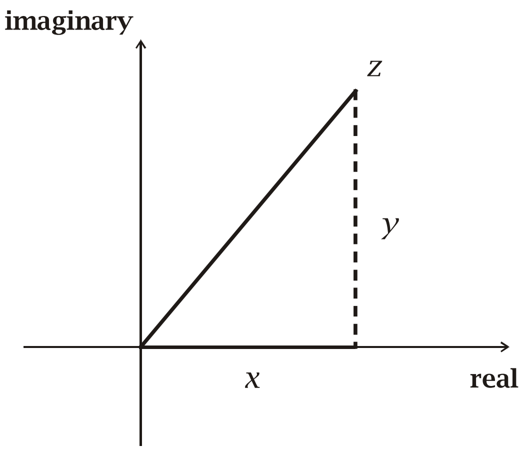
Complex Numbers and the Argand plane

The Argand Diagram

The *Argand diagram* is a graphical representation of a complex number that enables us to go some way towards visualising what a complex number is. A complex number is an abstract entity with no physical counterpart, but we can represent a complex number as a point in a plane where one coordinate represents the real part of the number and the other coordinate represents the imaginary part. The plane is called the *Argand plane* and the diagram is called the *Argand diagram*.



Example (1)

Plot the complex numbers

$z_1 = 1 + 2i$

$z_2 = -2 + i$

$z_3 = z_1 + z_2$

$z_4 = z_2 - z_1$

on the Argand plane

Solution

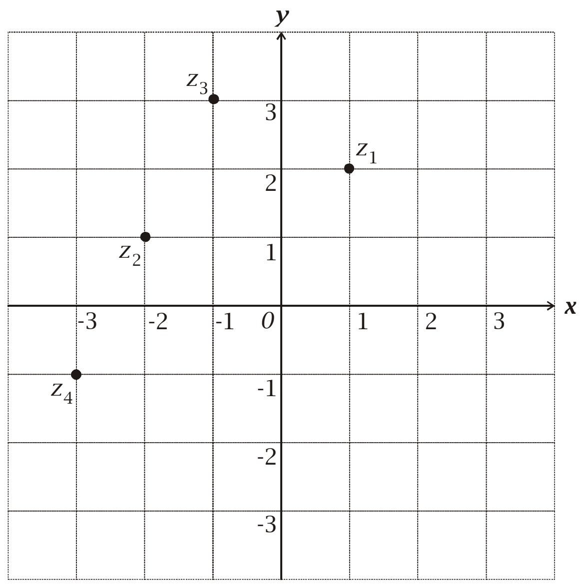
$z_1 = 1 + 2i$

$z_2 = -2 + i$

$z_3 = z_1 + z_2 = (1 + 2i) + (-2 + i) = -1 + 3i$

$z_4 = z_2 - z_1 = (-2 + i) - (1 + 2i) = -3 - i$

On the Argand diagram these points are



Real and imaginary parts

The complex number $z = x + iy$ has a real part x and an imaginary part y. This shows that it is a two dimensional object. Since it has two dimensions it is, in fact, a vector. The set of all complex numbers is denoted by the symbol $\mathbb{C}$ and is a two-dimensional space of vectors. It is useful to have functions that pick out the real and imaginary parts of a complex number. These functions called the *real* and *imaginary* parts of the complex number, z.

If $z = x + iy$
$\text{Re}(z) = x$
$\text{Im}(z) = y$

$\text{Re}(z)$ is read, "the real part of z".
$\text{Im}(z)$ is read, "the imaginary part of z".

Example (2)

Find $\text{Re}(z)$ and $\text{Im}(z)$ where $z = -2 - 4i$

Solution

$\text{Re}(z) = -2$ $\qquad$ $\text{Im}(z) = -4$

The functions $\text{Re}(z)$ and $\text{Im}(z)$ are also called *projection* functions because they create a one-dimensional image of a two-dimensional object. The Argand diagram makes it clear that if two complex numbers, z and w, have the same real and imaginary parts, then they must be the same number. This is written formally

If $\quad \mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$
then $\quad z = w$
and conversely

The phrase "and conversely" means that the result works the other way around; that is

If $\quad z = w$
then $\quad \mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$

Another way of writing this is

$\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$ **iff** $z = w$

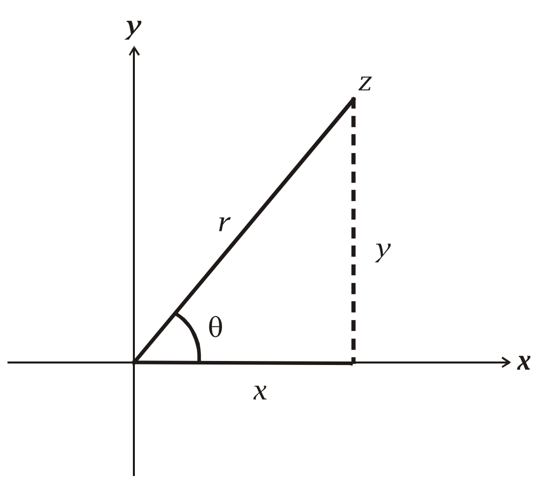
where **iff** is short for "if, and only if". There is also a symbolic way of writing this

$$\mathrm{Re}(z) = \mathrm{Re}(w) \text{ and } \mathrm{Im}(z) = \mathrm{Im}(w) \quad \Leftrightarrow \quad z = w$$

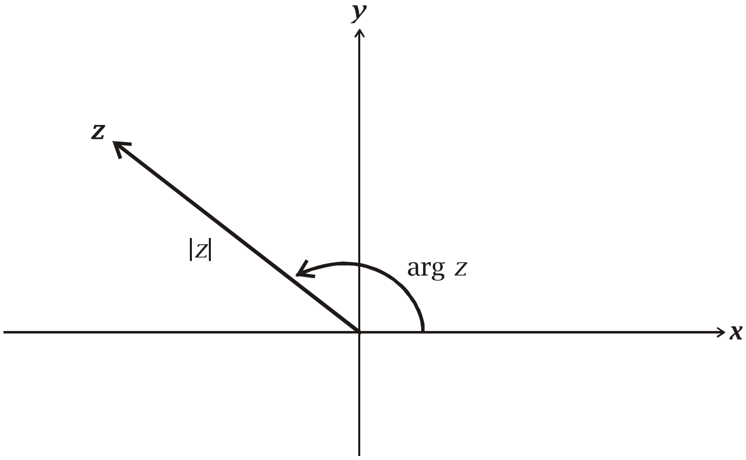
They all mean the same thing, namely, that the left hand side is interchangeable with the right-hand side.

The polar coordinates of a complex number

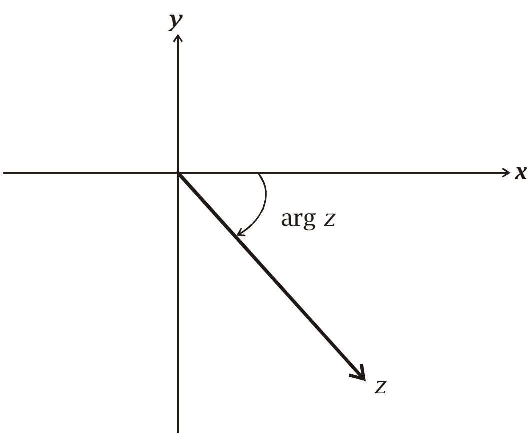
Since a complex number is a vector, it is possible to specify the locus (position) of a complex number in terms of an angle and a range. In other words, there is a polar representation of a complex number.



The polar representation is denoted $[r, \theta]$, where r = the distance of the complex number z from the origin in the Argand plane. This distance is also called the modulus of the complex number z, and is denoted $|z|$. That is $r = |z|$. $\theta =$ the angle at which z lies measured relative to the real axis, and taken in a clockwise sense. This angle is called the argument of the complex number z and is also denoted arg z.



The argument of z is often measured in radians. As the diagram indicates, the polar representation of a complex number is not unique since the complex number with argument θ is the same as the complex number with argument $\theta + 2\pi$ radians. This usually does not cause a difficulty, and can be a definite advantage, but if necessary we can create a unique representation of each complex number by restricting the domain to the principal value of the argument. This domain is usually specified as $[0, 2\pi)$. The positive direction for the argument is in an anti-clockwise direction. Complex numbers can have negative arguments – angles measured from the real axis (the x-axis) in a *clockwise* sense.



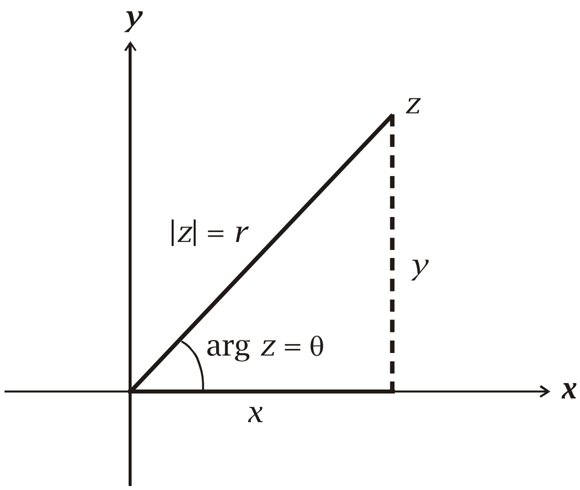
In this diagram, the argument is negative. The polar representation of a complex number is placed in square brackets

$[r, \vartheta] = [\,|z|, \arg z\,]$

in order to distinguish it from the Cartesian representation, in curved brackets

$(x, y) = x + iy$

One may ask – why have two forms of representation of the same complex number? As ever, mathematics is a language and it is useful to have different ways of looking at the same object. It is easier to add complex numbers in Cartesian form, and easier to multiply them in polar form. Since we have two sets of different representations, we need ways of converting between them.



To convert from Cartesian to polar coordinates

Given

$z = (x, y) = x + iy$

then

$|z| = r = \sqrt{x^2 + y^2}$ $\qquad$ $\arg z = \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Conversely, to convert from polar to Cartesian coordinates

Given

$z = [r, \theta] = [\,|z|, \arg z\,]$

then

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

Example (3)

Find the polar coordinates of $z = 1 + \sqrt{3}i$ and plot both polar and Cartesian coordinates of z on the Argand diagram.

Solution

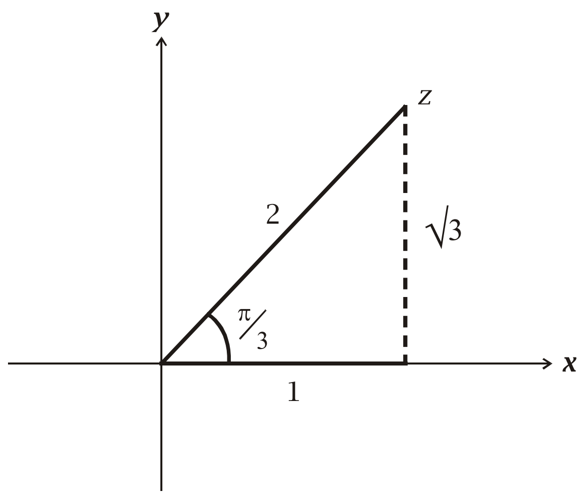
$z = 1 + \sqrt{3}i$

$x = 1 \qquad y = \sqrt{3}$

$|z| = r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$

$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$z = \left[2, \frac{\pi}{3}\right]$



Conjugate roots

If $z = x + iy$ then the complex conjugate is $z^* = x - iy$. Sometimes the symbol $\bar{z}$ is used to denote the complex conjugate. The roots of a quadratic equation, if they are complex roots, are complex conjugates. This follows from the quadratic formula

$az^2 + bz + c = 0$

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac < 0$ then z is complex. Then if

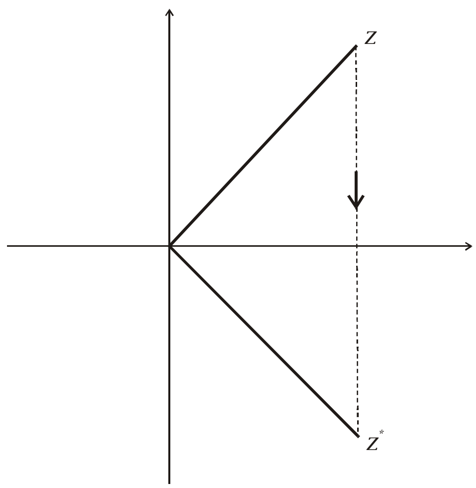
$z_1 = \frac{-b}{2a} + \left(\frac{\sqrt{4ac - b^2}}{2a}\right)i$ is one root,

then, $z_2 = z_1^* = \frac{-b}{2a} - \left(\frac{\sqrt{4ac - b^2}}{2a}\right)i$

is the other.

Representation of the complex conjugate in the Argand plane

The definition of the conjugate makes it clear that z^* is the reflection of z in the x-axis.



The complex number and its conjugate

Trigonometric form of a complex number

Given a complex number in polar form

$z = [r, \theta] = [\,|z|, \arg z\,]$

and since

$x = r\cos\theta \qquad y = r\sin\theta$

we can represent the complex number by a trigonometric form

$z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta)$

This gives us six ways of representing the same complex number summarised by just one line.

$z = x + iy = (x, y) = [r, \theta] = [\,|z|, \arg z\,] = r(\cos\theta + i\sin\theta)$

The ability to transform between these various representations lies at the heart of the subject.

Example (4)

Find the trigonometric form of the polar complex number $z = \left[2, \frac{\pi}{3}\right]$.

Solution

$$z = \left[2, \frac{\pi}{3}\right] = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$