Compound-angle Formulae

Prerequisites

You should know that

$\sin(-A) = -\sin A$

$\cos(-A) = \cos A$

$\tan(-A) = -\tan A$

You should be able to prove all of the following trigonometric identities.

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

(2) $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$

(3) $\sin^2\theta + \cos^2\theta \equiv 1$

(4) $\tan^2\theta + 1 \equiv \sec^2\theta$

(5) $1 + \cot^2\theta \equiv \text{cosec}^2\theta$

Example (1)

Given $\sin^2\theta + \cos^2\theta \equiv 1$ prove $\tan^2\theta + 1 \equiv \sec^2\theta$

Solution

$\sin^2\theta + \cos^2\theta \equiv 1$

On dividing both sides by $\cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$\tan^2\theta + 1 \equiv \sec^2\theta$

You should be able to apply these identities to solve trigonometric equations.

Example (2)

Solve

$\text{cosec}^2 x = 4\cot x - 2$ for $0 \le \theta \le 360°$

Solution

$$4\cot x - \operatorname{cosec}^2 x - 2 = 0$$

$$4\cot x - \left(1 + \cot^2\theta\right) - 2 = 0 \qquad \left[\operatorname{cosec}^2 x \equiv 1 + \cot^2 x\right]$$

$$-\cot^2\theta + 4\cot x - 3 = 0$$

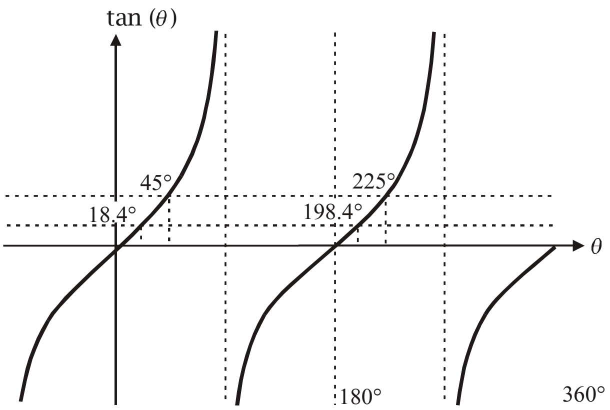
$$\cot^2\theta - 4\cot x + 3 = 0$$

$$\frac{1}{\tan^2\theta} - \frac{4}{\tan\theta} + 3 = 0 \qquad \left[\cot x = \frac{1}{\tan x}\right]$$

$$1 - 4\tan\theta + 3\tan^2\theta = 0$$

$$\left(3\tan\theta - 1\right)\left(\tan\theta - 1\right) = 0$$

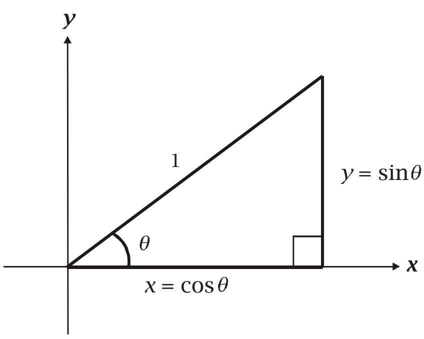
$$\tan\theta = \frac{1}{3} \quad \text{or} \quad \tan\theta = 1$$



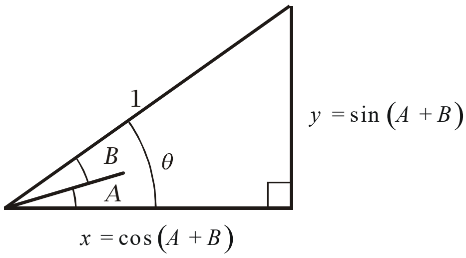
$$\theta = 18.4°,\ \ 45°,\ \ 198.4° \ \text{ or } \ 225° \ \left(\text{nearest } 0.1°\right) \ \text{ for } 0 \le \theta \le 360°$$

Compound-angle formulae

The trigonometric functions are defined by the ratios of the sides in the following triangle.



Suppose now that the angle θ in this diagram is divided in two, so that $\theta = A + B$ is a *compound angle.*



The adjacent and opposite sides of this triangle are $x = \cos(A+B)$ and $y = \sin(A+B)$ respectively. We would like to be able to find values for $x = \cos(A+B)$ and $y = \sin(A+B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$. Let us start by **disproving** one conjecture.

Example (3)

(*a*) By substituting values for $\cos A$ and $\cos B$ show that the *conjecture*

$$\cos(A+B) \equiv \cos A + \cos B$$

is **false**.

(*b*) In the expression $\cos(A+B) \equiv \cos A + \cos B$ what does the equivalence sign $(\equiv)$ mean?

(*c*) Given

$$\cos\left(A + \frac{\pi}{4}\right) = \cos A + \cos\left(\frac{\pi}{4}\right) \qquad 0 < A < 2\pi$$

By substituting $A = \left(\frac{3}{2}\pi\right)$ radians into the left and right-hand sides of this expression find a value of A in the interval $0 < A < 2\pi$ that makes this identity true.

Solution

(*a*) Let $A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$

Then $\cos A = \frac{1}{2}$ $\qquad \cos B = \frac{\sqrt{3}}{2}$ $\qquad A + B = \pi$ $\qquad \cos(A+B) = -1$

On substitution into

$$\cos(A+B) \equiv \cos A + \cos B$$

The left-hand side is

$$\cos(A+B) = \cos(\pi) = -1$$

The right-hand side is

$$\cos A + \cos B = \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} \neq -1$$

These are *not equal* so the conjecture

$$\cos(A+B) \equiv \cos A + \cos B$$

is *false*. Note: almost any two angles A and B would suffice to prove that this conjecture is false.

(*b*) This question is concerned with the distinction between *identities* involving the equivalence sign $(\equiv)$ and *equations* involving the equals sign $(=)$. In the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$ the sign $\equiv$ denotes equivalence between functions, and represents the idea that both sides of the identity are true for *all values* of the functions so that that the left-hand side, $\sin^2\theta + \cos^2\theta$, is *always* equal to the right-hand side, 1, whatever the value of the angle, θ. The symbol = represents an equality between numbers. Thus $\equiv$ stands for a relationship between functions, and = stands for a relationship between numbers.

(*c*) In this solution the angle A is measured in radians. We are given $\cos\left(A + \frac{\pi}{4}\right) = \cos A + \cos\left(\frac{\pi}{4}\right)$.

Substituting $A = \left(\frac{3}{2}\pi\right)$ we get

$$\text{LHS} = \cos\left(A + \frac{\pi}{4}\right) = \cos\left(\frac{3}{2}\pi + \frac{1}{4}\pi\right) = \cos\left(\frac{7}{4}\pi\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\text{RHS} = \cos A + \cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{3}{2}\pi\right) + \cos\left(\frac{\pi}{4}\right) = 0 + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \text{LHS}$$

So *the particular value* of A that makes the equation

$$\cos\left(A + \frac{\pi}{4}\right) = \cos A + \cos\left(\frac{\pi}{4}\right)$$

true in the interval $0 < A < 2\pi$ is $A = \frac{3}{2}\pi$.

This question shows that the most obvious guess at a formula for $\cos(A+B) \equiv ?$ is false, and reminds you of the important distinction between an identity and an equation. The formula for $\cos(A+B) \equiv ?$ is an identity that must be true for *all* values of angles A and B and not just an equation that can be solved for particular values of A and B. The formula for $\cos(A+B) \equiv ?$ is in fact given by

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$

Example (4)

Given $A = \frac{\pi}{3}$ substitute

$(a)\ \ B = \frac{\pi}{4}$ $\qquad (b)\ \ B = \frac{\pi}{6}$ $\qquad (c)\ \ B = \pi$

and verify in each case that the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ is true.

Solution

Substituting $A = \frac{\pi}{3}$ the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ becomes

$$\cos\left(\frac{\pi}{3}+B\right) \equiv \cos\left(\frac{\pi}{3}\right)\cos B - \sin\left(\frac{\pi}{3}\right)\sin B$$

$$\equiv \frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B$$

The left-hand side of this is

$$\text{LHS} = \cos\left(\frac{\pi}{3}+B\right)$$

and the right-hand side is

$$\text{RHS} = \frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B$$

	LHS	**RHS**
	$\cos\left(\frac{\pi}{3}+B\right)$	$\frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B$
$(a)\ \ B = \frac{\pi}{4}$	$\cos\left(\frac{7}{12}\pi\right) = -2.588..$	$\frac{1}{2}\cos\left(\frac{\pi}{4}\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{4}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}} = -2.588...$
$(b)\ \ B = \frac{\pi}{6}$	$\cos\left(\frac{1}{2}\pi\right) = 0$	$\frac{1}{2}\cos\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}-\sqrt{3}}{4} = 0$
$(c)\ \ B = \pi$	$\cos\left(\frac{4}{3}\pi\right) = -\frac{1}{2}$	$\frac{1}{2}\cos(\pi) - \frac{\sqrt{3}}{2}\sin(\pi) = \frac{-1-0}{2} = -\frac{1}{2}$

This shows that for **some** particular values of A and B the identity

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$

is true. However, it does *not* prove the identity.

Example (4) continued

Explain why we have not yet proven the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$.

Solution

The identity

$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$

must be true for **all values** of A and B. Our result that this expression is true for $A = \frac{\pi}{3}$ and (a) $B = \frac{\pi}{4}$ (b) $B = \frac{\pi}{6}$ and (c) $B = \pi$ could be just a fluke. We have not shown that there could be no counter-example to the truth of this statement.

Proving the double-angle formulae

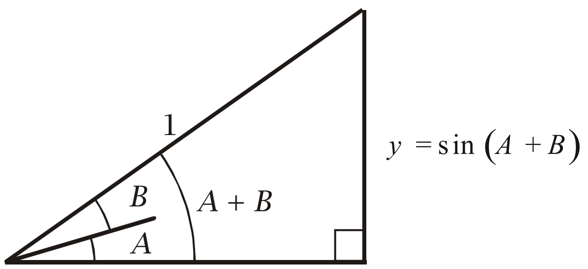
Two double-angle formulae are

(1) $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$

(2) $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

We will now prove the second of these, and invite you to prove the first by imitation of our proof.

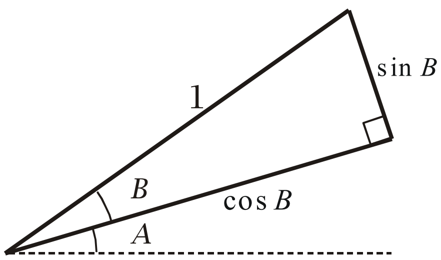
Let us begin by reminding ourselves of what we are trying to prove.



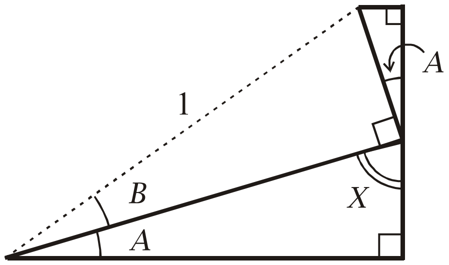
In the diagram above we are trying to establish that the side marked y here is also given by

$y = \sin A \cos B + \cos A \sin B$ whatever A and B.

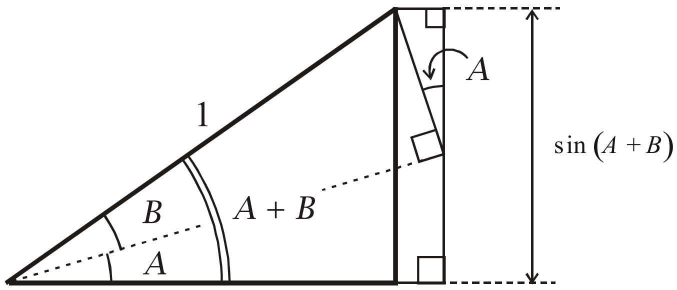
In the next diagram we interpret the meaning of $\cos B$ and $\sin B$.



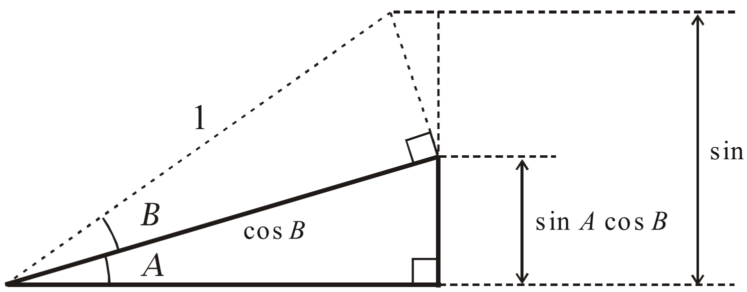
Now examine the following diagram.



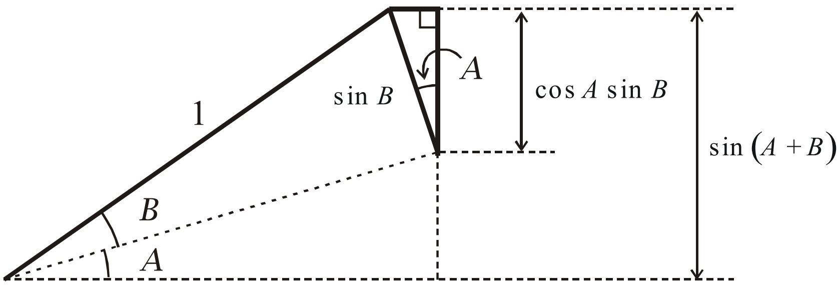
In this diagram there are two triangles where an angle has been marked by A. These two angles marked A are equal. This is because in both triangles where this angle is included we have $A + X = 90°$ where X is another angle as indicated in the diagram. The following diagram reminds us of what $\sin(A+B)$ means.



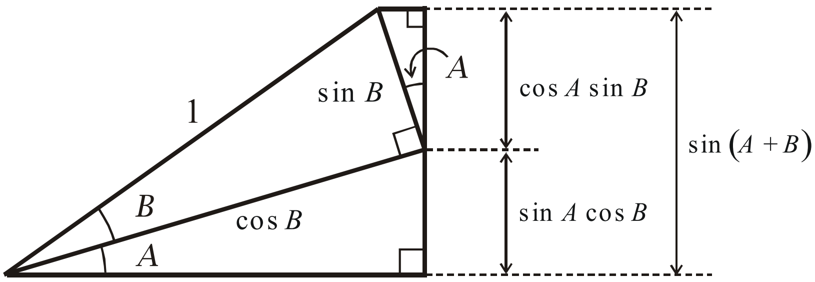
That is, $\sin(A+B)$ is the length of the side opposite the double angle $A+B$. The following diagram demonstrates that the side opposite the angle A in the triangle with hypotenuse of length $\cos B$ has length $\sin A \cos B$.



The next diagram demonstrates that the side adjacent to the angle A in the triangle with hypotenuse $\sin B$ has length $\cos A \sin B$.



Thus overall the side with length $y = \sin(A+B)$ shown in the following diagram



is also given by $y = \sin A \cos B + \cos A \sin B$. Furthermore, this argument does not depend on particular values of the angles *A* and *B*, but is true for *all* values of *A* and *B*. Therefore, we have established the identity

$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$.

Observe in this last line the switch from the equals $(=)$ to the equivalence symbol $(\equiv)$. This is valid because in the course of the proof we establish

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

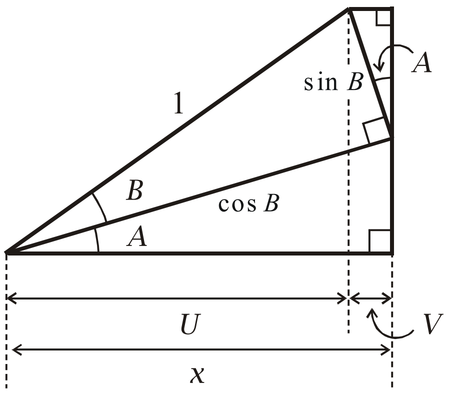
for **any** arbitrarily chosen angles *A* and *B*; and so we can finally conclude

$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

for **all** angles *A* and *B*.

Example (5)

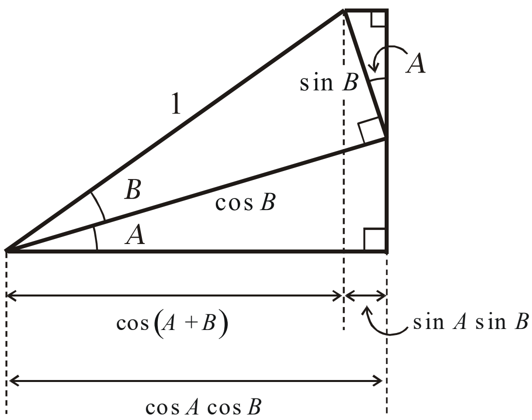
(*a*) In the following diagram find an expression for x in terms of $\cos(A+B)$ and expressions for U and V in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$.



(*b*) Hence prove the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$.

Solution

(*a*)



$x = \cos(A+B)$
$V = \cos A \cos B$
$U = \sin A \sin B$

(*b*) We have for any arbitrarily chosen angles *A* and *B*

$x = V - U$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$

But this argument does not depend on the particular values of the angles *A* and *B*. Hence for all *A* and *B*

$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$

More formulae from existing ones

We can now establish further double-angle formulae on the basis of the two we already have.

(1) $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$

(2) $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

We will add to these.

(3) $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$

Proof

This is proven by substituting –B for B in the formula for $\sin(A+B)$.

$$\sin(A-B) \equiv \sin A\cos(-B) + \cos A\sin(-B)$$
$$\equiv \sin A\cos B - \cos A\sin B$$

This follows from the identities

$\sin(-A) \equiv -\sin A$ — $\sin A$ is an odd (anti-symmetric) function

$\cos(-A) \equiv \cos A$ — $\cos A$ is an even (symmetric) function

(4) $\sin(2A) = 2\sin A\cos A$

Proof

This is proven by substituting A for B in the formula for $\sin(A+B)$.

$$\sin(A+A) \equiv \sin A\cos A + \cos A\sin A$$
$$\sin(2A) \equiv 2\sin A\cos A$$

(5) $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$

Example (6)

Prove this formula by substituting –B for B in the formula for $\cos(A+B)$.

Solution

$$\cos(A-B) \equiv \cos A\cos(-B) - \sin A\sin(-B)$$
$$\equiv \cos A\cos B + \sin A\sin B$$

(6) Formulae for $\cos(2A)$

Proof

On substitution of A for B in $\cos(A+B)$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2\cos^2 A - 1 \qquad [\sin^2 A + \cos^2 A \equiv 1]$$
$$\equiv 1 - 2\sin^2 A$$

(7) The preceding formula can be rearranged to give an expression for $\cos^2 A$.

$$\cos^2 A \equiv \frac{\cos 2A + 1}{2}$$

Proof

$$\cos 2A \equiv 2\cos^2 A - 1$$
$$2\cos^2 A \equiv \cos 2A + 1$$
$$\cos^2 A \equiv \frac{\cos 2A + 1}{2}$$

This formula is useful when integrating the expression $\cos^2 A$.

$$\begin{aligned}\int \cos^2 A\, dA &\equiv \int \frac{1}{2}\cos 2A\ dA + \int \frac{1}{2} dA \\ &\equiv \frac{\sin 2A}{4} + \frac{A}{2} + c\end{aligned}$$

Remark

The identities

$$\sin(2A) \equiv 2\sin A \cos A$$

$$\begin{aligned}\cos 2A &\equiv \cos^2 A - \sin^2 A \\ &\equiv 2\cos^2 A - 1 \\ &\equiv 1 - 2\sin^2 A\end{aligned}$$

are called the *double-angle formulae*.

Example (7)

By substituting $x = 2A$ into the double angle formulae above find the *half-angle formulae*.

Solution

$$\sin x \equiv 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\begin{aligned}\cos x &\equiv \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \\ &\equiv 2\cos^2\left(\frac{x}{2}\right) - 1 \\ &\equiv 1 - 2\sin^2\left(\frac{x}{2}\right)\end{aligned}$$

Using the compound-angle formulae

We can now apply these compound-angle formulae to a variety of examples.

Example (8)

By writing $\sin 105^\circ$ as $\sin(A+B)$ where $A = 60^\circ$ find the exact value of $\sin 105^\circ$ expressing your answer in surd form.

Solution

$\sin 105^\circ = \sin(60^\circ + 45^\circ)$

On substitution into

$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

$$\begin{aligned}\sin(105^\circ) &= \sin(60^\circ + 45^\circ)\\ &= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ\\ &= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}\times\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

Example (9)

Find all the values of θ in the range $0 \le \theta \le 360^\circ$ satisfying

$\sin 2\theta = \sqrt{2}\cos\theta$

Solution

$\sin 2\theta = \sqrt{2}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$

$\cos\theta = 0 \quad \text{or} \quad 2\sin\theta = \sqrt{2}$

$\cos\theta = 0 \qquad \Rightarrow \qquad \theta = 90^\circ \quad \text{or} \quad \theta = 270^\circ$

$2\sin\theta = \sqrt{2} \qquad \Rightarrow \qquad \sin\theta = \frac{1}{\sqrt{2}}$

$\qquad \Rightarrow \qquad \theta = 45^\circ \quad \text{or} \quad \theta = 135^\circ$

$\theta = 45^\circ, 90^\circ, 135^\circ \quad \text{or} \quad 270^\circ$

Example (10)

Find all the values of θ in the range $0 \le \theta \le 360^\circ$ satisfying

$\cos(A - 30^\circ) = \frac{1}{2}\cos A$

Solution

$\cos(A-30°)=\frac{1}{2}\cos A$

$\cos A\cos 30°+\sin A\sin 30°=\frac{1}{2}\cos A$

$\frac{\sqrt{3}}{2}\cos A+\frac{1}{2}\sin A=\frac{1}{2}\cos A$

$\left(\sqrt{3}-1\right)\cos A=-\sin A$

$\frac{\sin A}{\cos A}=1-\sqrt{3}$

$\tan A=1-\sqrt{3}$

$A=143.8°$ or $323.8°$ (nearest $0.1°$)

Example (11)

Find all the values of θ in the range $0\le\theta\le 360°$ satisfying

$1+2\cos(2x)=3\cos x$

Solution

$1+2\cos(2x)=3\cos x$

$1+2\left(2\cos^2 x-1\right)=3\cos x$

$1+4\cos^2 x-2=3\cos x$

$4\cos^2 x-3\cos x-1=0$

$(4\cos x+1)(\cos x-1)=0$

$\cos x=-\frac{1}{4}$ or $\cos x=1$

$\cos x=-\frac{1}{4} \quad\Rightarrow\quad x=104.5°$ or $255.5°$ (nearest $0.1°$)

$\cos x=1 \quad\Rightarrow\quad x=0°$ or $x=360°$

$x=0°,\ 75.5°, 284.5°$ or $360°$

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