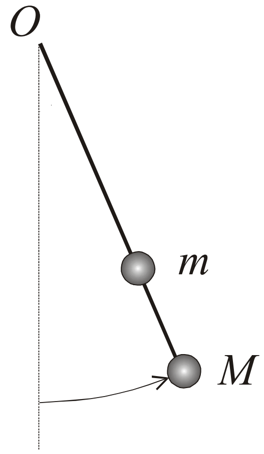
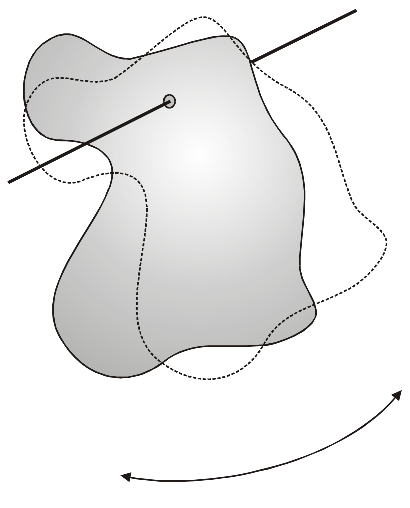
Compound pendulum

A compound pendulum is a body made of two or more masses connected together and swinging freely about a smooth horizontal axis. The simplest case would be when just two masses are connected.



However, there may be more than two masses, and a compound pendulum could be made of an irregular lamina.



This can be considered to be a compound pendulum, since the lamina can be thought of as being made up of a large number of smaller segments, each with their own mass.

The problem of the compound pendulum is essentially to find its period of oscillation (or equivalently, to find its angular frequency). This problem is solved by reducing the compound pendulum to a simple pendulum to which it is equivalent. This is done through energy considerations as shown in the example that follows.

The crucial things to remember are

(i)

Gravitational potential energy is found from the usual formula

$u = mgh$

where h is the height of the centre of mass above some reference point.

(ii)

Kinetic energy is given by

$E_k = \frac{1}{2} I\dot{\theta}^2$

where I is the moment of inertia of the object, and $\dot{\theta} = \frac{d\theta}{dt}$ is the angular velocity that the line joining the centre of mass to the pivot point makes with the vertical.

The point of this type of problem is captured by the observation that the expression for the moment of inertia of an object, I, replaces mass in the usual formula for *linear* kinetic energy. Since the moment of inertia has replaced mass, we need to be able to find it. In problems of this type we usually start with the standard moments of inertia for objects such as discs, rods and rectangular laminas, and derive the moment of inertia for the composite body using the parallel axis theorem. Hence, we need to keep the parallel axis theorem also in our minds:

(iii)

The parallel axis theorem states that, supposing the moment of inertia of a body M about an axis passing through its centre of mass is Mk^2, then its moment of inertia about an axis parallel to this first axis but at a distance d from it is $M\left(k^2 + d^2\right)$.

(iv)

Since the system is assumed to be frictionless, total energy is conserved. Hence,

$E = E_k + U = \text{constant}$

That is,

$\frac{1}{2} I\dot{\theta}^2 + mgh = \text{constant}$

On differentiating

$I\dot{\theta}\ddot{\theta} + mg\sin\theta\dot{\theta} = 0$

We cancel through $\dot{\theta}$ and rearrange to give the equation of motion for the compound pendulum

$\ddot{\theta} = -k\sin\theta$

If assume that because the amplitude of the oscillation is small then $\sin\theta \approx \theta$, then this equation reduces to the equation for simple harmonic motion

$\ddot{\theta} = -k\theta$

with solution

$\theta(t) = \sin(\omega t + \alpha)$

where $\omega = \sqrt{k}$ is the angular frequency, and α is the phase shift.

This assumes that the aim of the problem is to derive the equation of simple harmonic motion from the data presented in the question. However, other questions involving energy conversions can be set, but these follow the same principle that total energy is conserved.

Just a note about the use of the symbol ω. This symbol is used here to denote angular frequency. This is the angle swept out per unit time, and is a constant for the particular oscillation described by the simple or compound pendulum (or object under simple harmonic motion generally). This symbol is also used to denote angular velocity

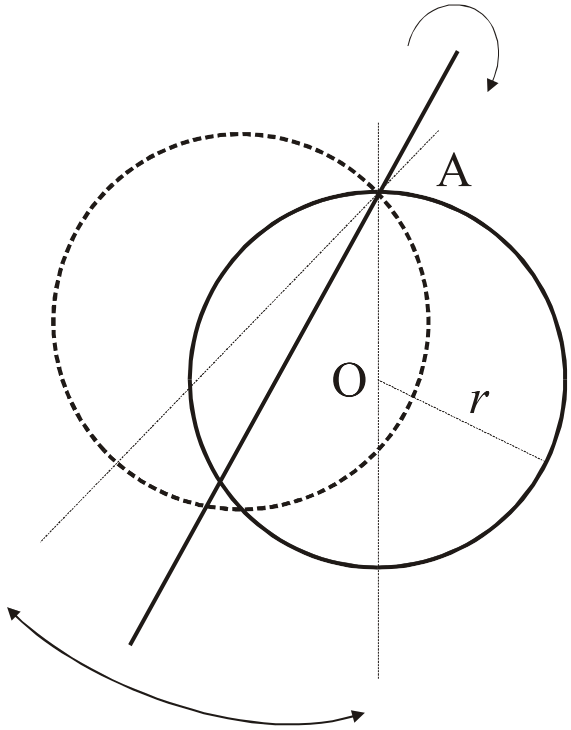
$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

which is a variable that depends on time, and is maximum as the pendulum passes through the vertical and zero at the points of maximum angular amplitude. Because of the possible confusion of having the same symbol to designate two different mathematical concepts, we advise you in this context to use $\dot{\theta}$ to denote the angular velocity, and ω to denote the angular frequency.

Example

(i) A disc of mass *m* and radius *r* is suspended from a point *A* on its rim. The centre of the disc is the point *O*. Initially, the disc is hanging so that the line *OA* is vertical. Assuming that the point of contact is frictionless, and that the disc can be treated as a uniform lamina, show that when the line *OA* is displaced slightly from the vertical by an angle α that the disc oscillates with simple harmonic motion, and find an expression for its angular frequency.

(*ii*) The disc is now stopped and positioned so that the line *OA* makes an angle of $\pi/2$ with the vertical. It is then released. Find the angular velocity of the disc as *OA* first becomes vertical.



Solution

To solve this problem we must first find an expression for the moment of inertia of the disc about the pivot point *A*.

The parallel axis theorem states that, supposing the moment of inertia of a body *M* about an axis passing through its centre of mass is Mk^2, then its moment of inertia about an axis parallel to this first axis but at a distance d from it is $M\left(k^2 + d^2\right)$.

The moment of inertia of the disc is

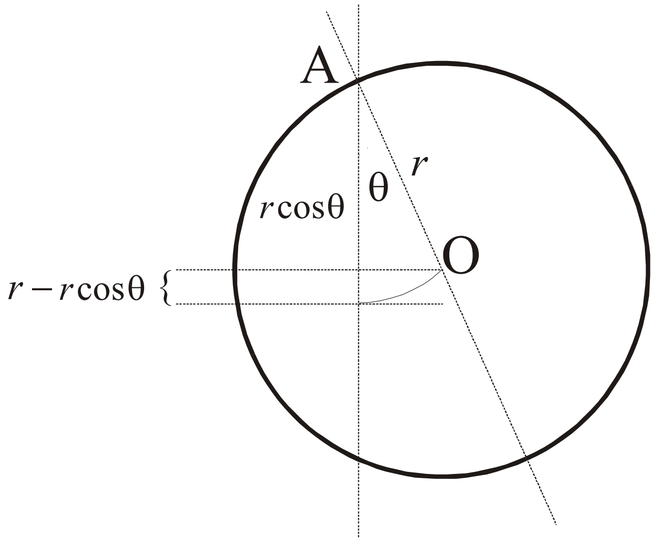
$$I_{disc} = \tfrac{1}{2}mr^2$$

The disc is at a distance *r* from the pivot point, *A*. Hence

$$I_A = \tfrac{1}{2}mr^2 + mr^2 = \tfrac{3}{2}mr^2$$

Let $\dot{\theta} = \dfrac{d\theta}{dt}$ be the angular velocity , then the kinetic energy of the disc when it makes an angle θ with the vertical is given by

$$E_K = \tfrac{1}{2}I^2\left(\dot{\theta}\right)^2$$



At this point the centre of the disc, which is its centre of gravity, has risen $r - r\cos\theta$, so the gravitational potential energy of the disc is

$$U = mgh = mg\left(r - r\cos\theta\right)$$

Since the system is frictionless, total energy is conserved, so

$$E = E_K + U$$
$$E = \tfrac{1}{2}I\left(\dot{\theta}\right)^2 + mgr\left(1 - \cos\theta\right)$$

where E is a constant.

On differentiating both sides

$$I\dot{\theta}\ddot{\theta} + mgr\sin\theta\dot{\theta} = 0$$

(The derivative of E¸ the constant energy, is zero.) On dividing by $\dot{\theta}$ and rearranging we obtain

$$I\ddot{\theta} = -mgr\sin\theta$$

This is the equation of motion for the body. As already indicated, here

$$I = \tfrac{3}{2}mr^2$$

Hence,

$\frac{3}{2}mr^2\ddot{\theta} = -mgr\sin\theta$

$\ddot{\theta} = -\frac{2g}{3r}\sin\theta$

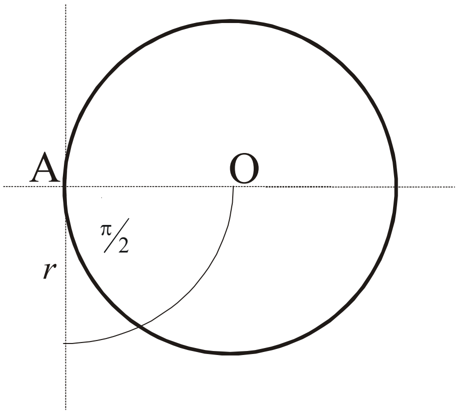
If the angle of displacement is small, then $\sin\theta \approx \theta$ hence

$\ddot{\theta} = -\frac{2g}{3r}\theta$

This is the equation of simple harmonic motion, and has angular frequency

$\omega = \sqrt{\frac{2g}{3r}}$

(ii)



The centre of mass is raised to an intial height r above its equilibrum position. It acquires mgr joules of gravitational potential energy.
As before the moment of inertia is
$I = \frac{3}{2}mr^2$
The kinetic energy of the disc is given by
$E_K = \frac{1}{2}I\dot{\theta}^2 = \frac{3}{4}mr\dot{\theta}^2$
The gravitational potential energy is converted to kinetic energy when the disc has falled so that the line OA becomes vertical. Hence
$mgr = \frac{3}{4}mr\dot{\theta}^2$

$\dot{\theta}^2 = \frac{4g}{3r}$

$\dot{\theta} = 2\sqrt{\frac{g}{3r}}$

Example (2)

A uniform rod, *AB*, has mass 6*m* and length 4*l*. It is suspended so that it can move freely about a smooth horizontal axis through *A*. A further particle of mass 4*m* is attached to it at *B*. From a position of hanging at rest in equilibrium, it is given an initial angular velocity of $\sqrt{g/3l}$. Find the maximum height to which the line *AB* rises.

Solution

The moment of inertia of a rod about an axis perpendicular to the rod and 2*l* distant from its centre of mass is given by

$$I = \frac{ML^2}{3}$$

Here the rod has a mass $M = 5m$ and a length, $L = 4l$, hence its moment of inertia is

$$I_{rod} = \frac{6m\left(2l\right)^2}{3} = 8ml^2$$

The moment of inertia of a particle at a distance *r* from an axis of rotation is given by

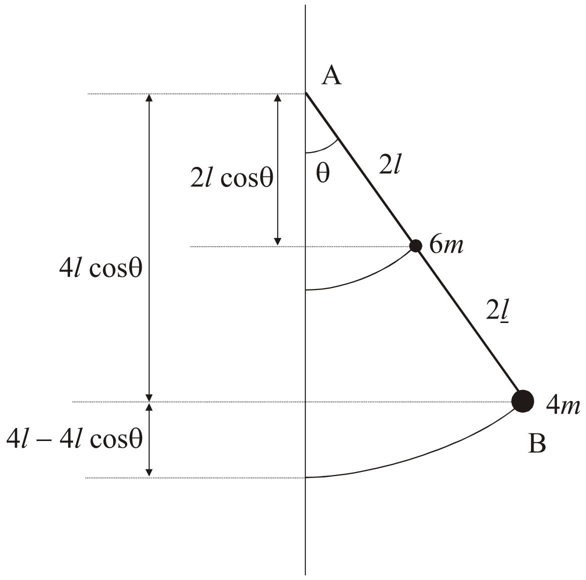
$$I = Mr^2$$

Here $r = 4l$ and $M = 4m$, hence

$$I_{particle} = 4m \times \left(4l\right)^2 = 64ml^2$$

The total moment of inertia for the rod and particle is, therefore

$$I = I_{rod} + I_{particle} = 8ml^2 + 64ml^2 = 72ml^2$$



Let the angle made by the rod *AB* with the vertical at the moment of instantaneous rest be θ. Then, when the rod has risen to this height, it has acquired $2l - 2l\cos\theta$ joules of gravitational potential energy. At the same time the particle has acquired $4l - 4l\cos\theta$ joules of gravitational potential energy. The total system has

$$U = 6mg\left(2l - 2l\cos\theta\right) + 4m\left(4l - 4l\cos\theta\right) = 28mgl\left(1 - \cos\theta\right)$$

joules of gravitational potential energy.

The initial kinetic energy is

$$K_E = \tfrac{1}{2}I\dot{\theta}^2 = \tfrac{1}{2}\times 72ml^2 \times \left(\sqrt{g/3l}\right)^2 = 12mgl$$

The kinetic energy is converted to gravitation potential energy, hence

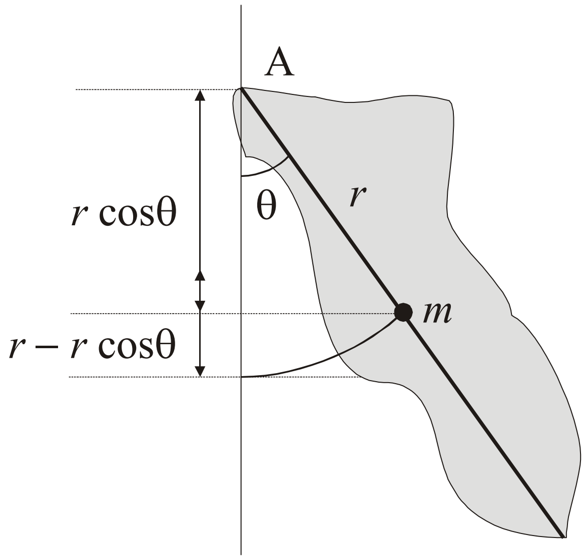
$$U = K_E$$

On substitution

$$28mgl\left(1 - \cos\theta\right) = 12mgl$$
$$-28\cos\theta = -16$$
$$\theta = \cos^{-1}\left(16/28\right) = 55.2^\circ\left(\text{to nearest } 0.1^\circ\right)$$

Compound pendulum and its equivalent simple pendulum

The method of solution in the first example would apply to any object of mass m with centre of mass distant r from the axis of rotation.



At time t the object will have risen $r - r\cos\theta$, where $\theta = \theta(t)$ is a function of time, so the gravitational potential energy of the disc is

$$U = mgh = mg(r - r\cos\theta)$$

Since the system is frictionless, total energy is conserved, so

$$E = E_K + U$$

$$E = \tfrac{1}{2} I\left(\dot{\theta}\right)^2 + mgr(1 - \cos\theta)$$

where E is a constant and I is the moment of inertia.

As before, on differentiating both sides

$$I\dot{\theta}\ddot{\theta} + mgr\sin\theta\dot{\theta} = 0$$

On dividing by $\dot{\theta}$ and rearranging we obtain

$$I\ddot{\theta} = -mgr\sin\theta$$

which is general the equation of motion for the compound pendulum.

If the angle of displacement is small, then $\sin\theta \approx \theta$ hence

$$\ddot{\theta} = -\frac{mgr}{I}\theta \qquad (1)$$

This is the equation of angular simple harmonic motion, and has angular frequency

$$\omega = \sqrt{\frac{mgr}{I}}$$

If we compare this with the equation for the simple pendulum

$$\ddot{\theta} = -\frac{g}{l}\theta$$

where *l* is the length of the simple pendulum, we can see that they are very similar.

If in equation (1) we set

$$L = \frac{I}{mr}$$

we see that it also takes the form

$$\ddot{\theta} = -\frac{g}{L}\theta$$

The quantity

$$L = \frac{I}{mr}$$

is called the *length of the equivalent simple pendulum*.

This means that if we know the mass, *m,* of the object, the distance, *r*, of the object's centre of mass from the axis of revolution, and the object's moment of inertia, we can find the object's equivalent length as a simple pendulum. On substitution into the simplified pendulum equation, we obtain its angular frequency, or equivalently, its period.

Alternatively, we obtain any one of the terms, *I, m,* and *r* given the angular frequency (or period) and two of the others. So we could "work backwards".

Example (3)

In the first example, find the length of the equivalent simple pendulum.

Solution

The solution to the first example was

$$\ddot{\theta} = -\frac{2g}{3r}\theta$$

which gives $L = \frac{3r}{2}$

Alternative, recall that the moment of inertia was

$$I = \tfrac{3}{2}mr^2$$

Hence, on substitution into

$$L = \frac{I}{mr}$$

we obtain

$$L = \frac{\tfrac{3}{2}mr^2}{mr} = \frac{3r}{2}$$