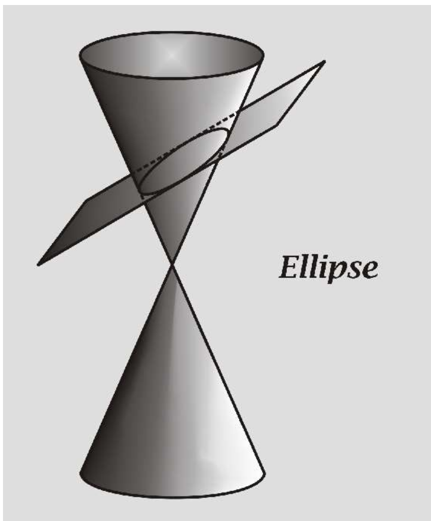
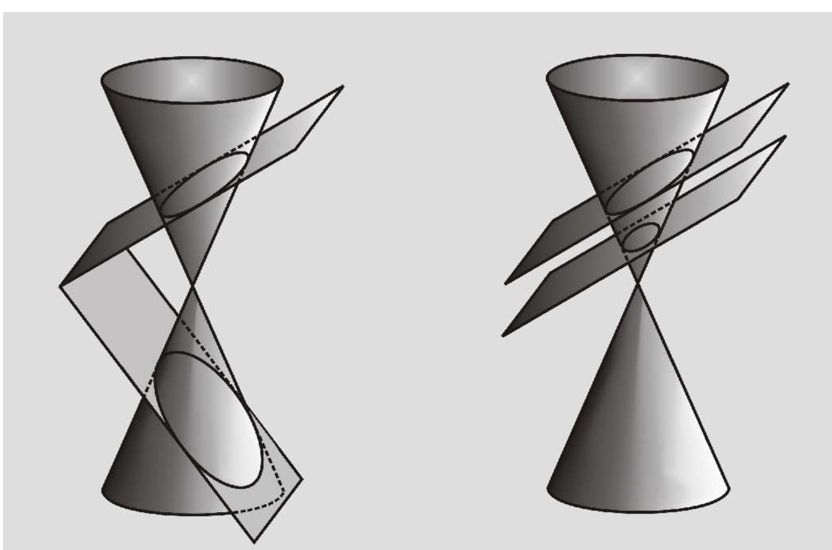
Conic sections

Conic sections

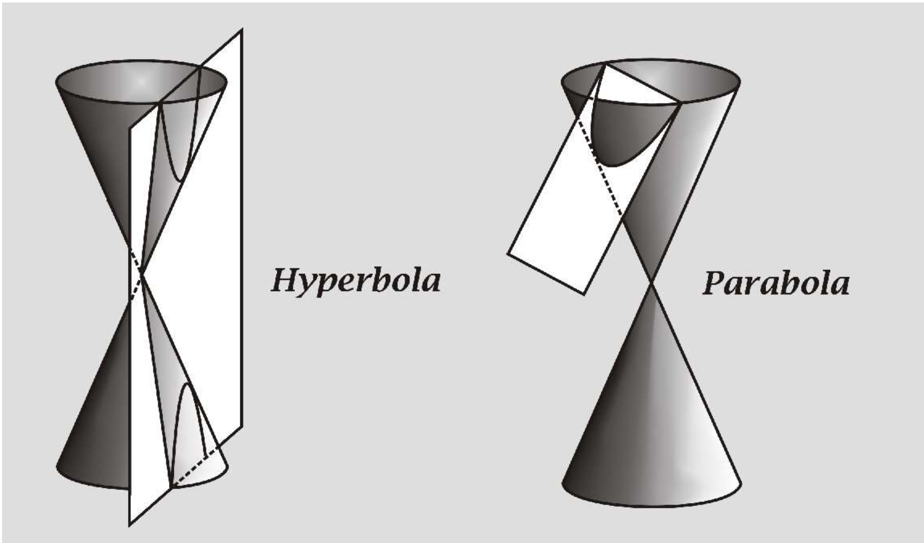
Conic sections are curves made by the intersection of a plane and a double cone. For example, the ellipse is made in the following way.



Different ellipses are formed either by rotation of the plane about a fixed line or by sliding the fixed line about which the plane is rotated up and down

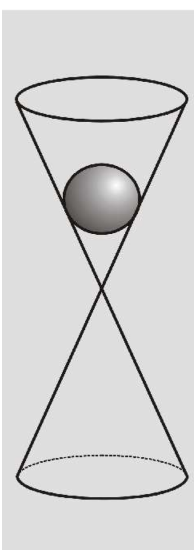


The hyperbola and parabola arises from other kinds of section of the double cone.

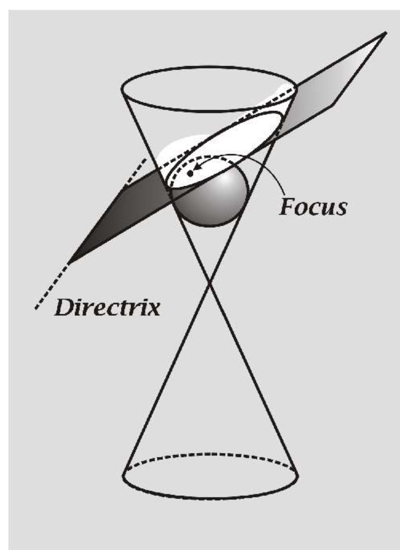


The focus-directrix property of conics

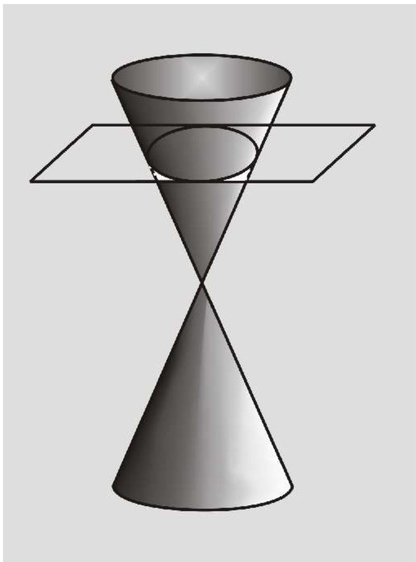
All conic sections have what is known as a focus-directrix property. This means that the curve is defined by reference to a line (the *directrix*) and a point (the *focus*). To show how this arises, firstly imagine dropping a sphere inside a cone.



We now draw a line outside the cone and construct a plane that uses this line as a hinge and intersects the cone in such a way that it just touches the sphere. The line is the *directrix,* and the point where the plane just touches the sphere is the *focus.* By adjusting the position of the directrix and the size of the sphere (and hence the position of the focus) we can define every kind of conic section – every kind of ellipse, hyperbola and parabola.



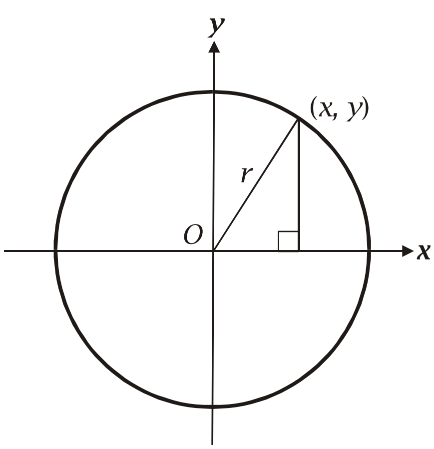
A circle is made by the intersection of a plane perpendicular to the central axis of the cone.



The circle

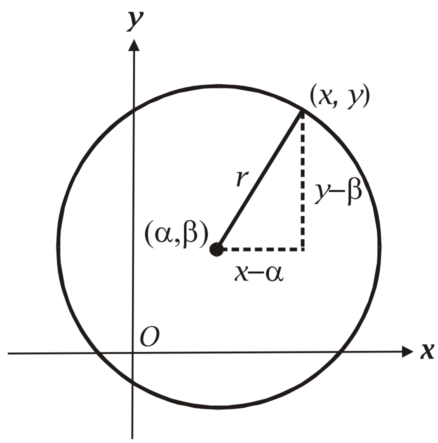
The Cartesian form of the circle is

$x^2 + y^2 = r^2$



If the centre of the circle is shifted to the point (α, β) then the equation becomes

$(x-\alpha)^2 + (y-\beta)^2 = r^2$



Multiplying out this expression gives

$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0$

The terms α, β and r are all constants, and it is usual to express this equation in the form

$x^2 + y^2 + 2gx + 2fy + c = 0$

By comparing the two forms we can see that the centre is at $(-g, -f)$ and the radius is $\sqrt{(g^2 + f^2 - c)}$

Example (1)

Find the Cartesian form of the equation of the circle with centre (–4, 3) and radius 5

Solution

$(x - (-4))^2 + (y - 3)^2 = 5^2$

$(x + 4)^2 + (y - 3)^2 = 5^2$

Example (2)

Find the centre and radius of the circle whose Cartesian equation is

$x^2 - 6x + y^2 + 12y + 29 = 0$

Solution

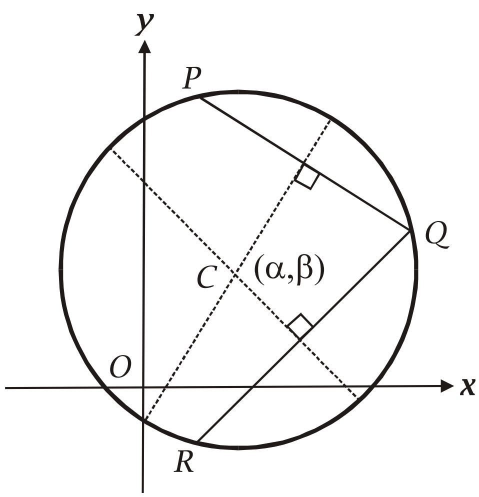
Begin by completing the squares for both the x and y terms

$x^2 - 6x + y^2 + 12y + 29 = 0$

$$\left(x^2-6x\right)+\left(y^2+12y\right)+29=0$$
$$\left(x^2-6x+(3)^2-(3)^2\right)+\left(y^2+12y+6^2-6^2\right)+29=0$$
$$(x-3)^2-9+(y+6)^2-36+29=0$$
$$(x-3)^2+(y+6)^2=16$$
$$(x-3)^2+(y+6)^2=4^2$$

So the circle has $(3,-6)$ and radius 4 .

The centre of a circle lies at the intersection of two perpendicular bisectors of cords joining points on the circle.



Example (3)

Find. the circle which passes through the points $(6,4)$, $(7,-3)$, $\left(1,-\sqrt{21}\right)$

Solution

The gradient of the cord joining $(6,4)$ to $(7,-3)$ is

$$m=\frac{\Delta y}{\Delta x}=\frac{y_1-y_2}{x_1-x_2}=\frac{4-(-3)}{6-7}=-7$$

Therefore, the gradient of the perpendicular to this cord is $1/7$

The midpoint of this cord is $(6.5,0.5)$

The equation of the perpendicular bisector of this cord is found by substitution into

$y=mx+c$

$y = \frac{1}{7}x + c$

$\frac{1}{2} = \frac{1}{7} \times \frac{13}{2} + c$

$c = \frac{1}{2} - \frac{13}{14} = \frac{7-13}{14} = -\frac{6}{14} = -\frac{3}{7}$

Hence $y = \frac{1}{7}x - \frac{3}{7}$ or $7y = x - 3$

The gradient of the cord joining $(7, -3)$ to $\left(1, -\sqrt{21}\right)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-3 - \left(-\sqrt{21}\right)}{7 - 1} = \frac{-3 + \sqrt{21}}{6}$$

Therefore, the gradient of the perpendicular to this cord is $\frac{6}{3 - \sqrt{21}}$

The midpoint of this cord is $\left(4, \frac{-3 - \sqrt{21}}{2}\right)$

The equation of the perpendicular bisector of this cord is found by

$y = mx + c$

$y = \frac{6}{3 - \sqrt{21}}x + c$

$\frac{-3 - \sqrt{21}}{2} = \left(\frac{6}{3 - \sqrt{21}}\right) \times 4 + c$

$c = \frac{-3 - \sqrt{21}}{2} - \frac{24}{3 - \sqrt{21}} = \frac{\left(-3 - \sqrt{21}\right)\left(3 - \sqrt{21}\right) - 48}{2\left(3 - \sqrt{21}\right)} = \frac{-9 + 21 - 48}{2\left(3 - \sqrt{21}\right)} = \frac{-36}{2\left(3 - \sqrt{21}\right)} = \frac{-18}{3 - \sqrt{21}}$

Hence $y = \frac{6}{3 - \sqrt{21}}x + \frac{-18}{3 - \sqrt{21}}$

Solving

$7y = x - 3 \qquad (1)$

$y = \frac{6}{3 - \sqrt{21}}x + \frac{-18}{3 - \sqrt{21}} \qquad (2)$

From (1) $\qquad x = 7y + 3$

In (2)

$y = \frac{6}{3 - \sqrt{21}}(7y + 3) + \frac{-18}{3 - \sqrt{21}}$

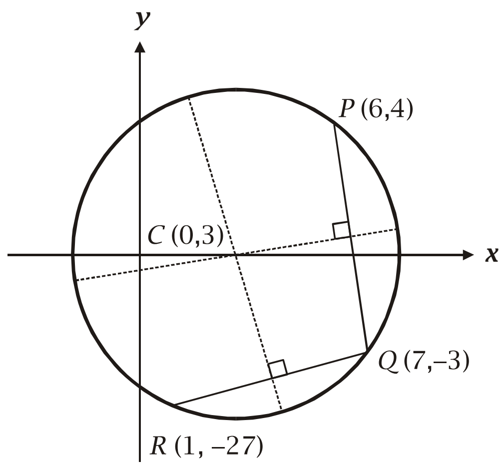
$y = \frac{42y}{3 - \sqrt{21}} + \frac{18}{3 - \sqrt{21}} - \frac{18}{3 - \sqrt{21}}$

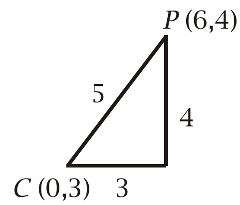
$y = \frac{42y}{3 - \sqrt{21}}$

$y = 0$

$x = 3$

Hence, the circle has centre $(3, 0)$





The radius of the circle is found by Pythagoras as

$$r = \sqrt{(6-3)^2 + (4-0)^2} = \sqrt{3^2 + 4^2} = 5$$

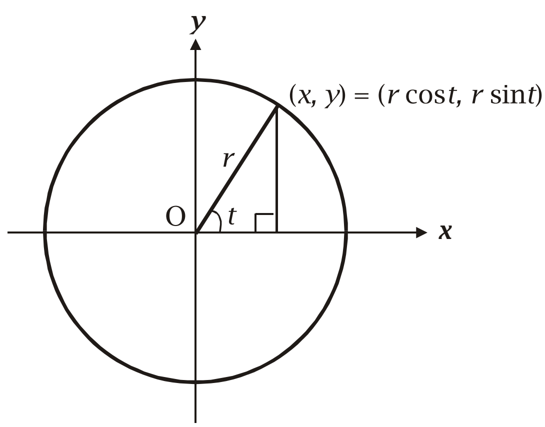
Parametric equation of the circle

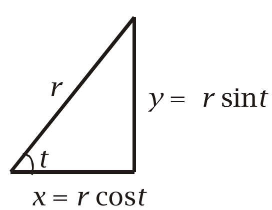
The circle, centred on the origin, has parametric equations

$x = a\cos t$

$y = a\sin t$

where $0 \le t < 2\pi$





Example (4)

A circle has parametric equations

$x = 2 + 3\cos t$

$y = 3\sin t$

$0 \le t < 2\pi$

(i) Find the cartesian equation of the circle.

(ii) Let P be the point where $t = \pi/4$. Find the equation of the normal to the circle at P.

Solution

(i) Since $x = 2 + 3\cos t$

$\cos t = \frac{x-2}{3}$

and since $y = 3\sin t$

$\sin t = \frac{y}{3}$

Substitution into $\sin^2 t + \cos t^2 t = 1$ gives

$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$x^2 - 4x + 4 + y^2 = 9$

Hence

$x^2 - 4x + y^2 - 5 = 0$

is the Cartesian equation of the circle.

(ii) $x = 2 + 3\cos t$ $\qquad$ $y = 3\sin t$

$\frac{dx}{dt} = -3\sin t$ $\qquad$ $\frac{dy}{dt} = 3\cos t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t}{-3\sin t} = -\cot t$

The gradient of the tangent is, therefore, $-\cot t$

Hence, the gradient of the normal is $m = \tan t$

When $t = \pi/4$, $x = 2 + 3\cos \pi/4 = 2 + \frac{3}{\sqrt{2}}$, $y = 3\sin \pi/4 = \frac{3}{\sqrt{2}}$

The equation of the normal is found by

$y = mx + c$

$y = (\tan t)x + c$

$y = \tan\left(\pi/4\right)x + c$

$y = x + c$

On substituting for x and y

$\frac{3}{\sqrt{2}} = \left(2 + \frac{3}{\sqrt{2}}\right) + c$

$c = \frac{3}{\sqrt{2}} - \left(2 + \frac{3}{\sqrt{2}}\right) = -2$

Hence, the equation of the normal is

$y = x - 2$

The parabola

The parabola has parametric equations

$x = at^2$ $y = 2at$ where a is a positive constant.

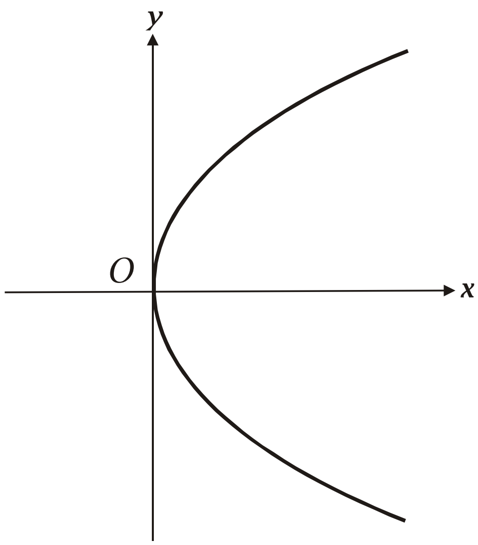
Eliminating t from this expression gives

$$t = \frac{y}{2a}$$

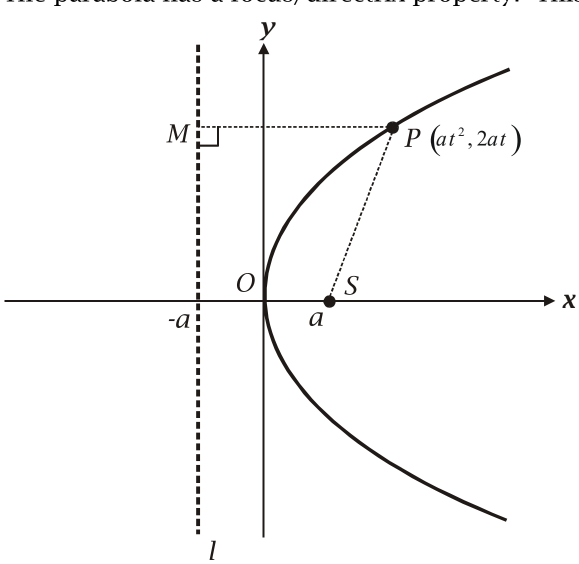
$$x = at^2 = a\left(\frac{y}{2a}\right)^2 = \frac{ay^2}{4a^2} = \frac{y^2}{4a}$$

Hence $y^2 = 4ax$

This gives the Cartesian equation of the parabola. Its graph is



The parabola has a focus/directrix property. This is illustrated by the following diagram



The line l runs parallel to the y-axis, with equation $x = -a$. This is the directrix. The point S lies at a on the x-axis and is called the *focus*. The line PM runs parallel to the x-axis – that is, it is perpendicular to the directrix. The parabola is defined by the ratio

$$\frac{PS}{PM} = 1$$

We can show that this is the case as follows.

$$PM = at^2 + a = a\left(1+t^2\right)$$

$$\begin{aligned} PS^2 &= \left(at^2 - a\right)^2 + \left(2at\right)^2 \\ &= a^2 - 2a^2t^2 + a^2t^4 + 4a^2t^2 \\ &= a^2 + 2a^2t^2 + a^2t^4 \\ &= a^2\left(1 + 2t + t^4\right) \\ &= a^2\left(1+t^2\right)^2 \end{aligned}$$

$$\therefore PS = a\left(1+t^2\right)$$

$$\therefore \frac{PS}{PM} = \frac{a\left(1+t^2\right)}{a\left(1+t^2\right)} = 1$$

Example (5)

(i) A parabola has focus $(3,0)$ and directrix $x = -3$. Find its equation in Cartesian form.

(ii) A parabola has Cartesian equation $y^2 = 16x$. Find its focus.

Solution

(i) The equation of the parabola in Cartesian form is

$y^2 = 4ax$ where $x = -a$ is the directrix and $(a,0)$ is the focus.

$\therefore y^2 = 12x$ is the Cartesian equation of this parabola.

(ii) Comparing $y = 12x$ with $y = 4ax$, then $a = 3$, so the focus is at $(3,0)$

The ellipse

The ellipse has parametric equations

$$x = a\cos t$$
$$y = b\sin t$$

where $0 \le t < 2\pi$

The Cartesian equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example (6)

Derive the Cartesian equation for the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from the parametric form

$x = a\cos t$
$y = b\sin t$
where $0 \le t < 2\pi$

Solution

Since $x = a\cos t$ then $\cos t = \frac{x}{a}$

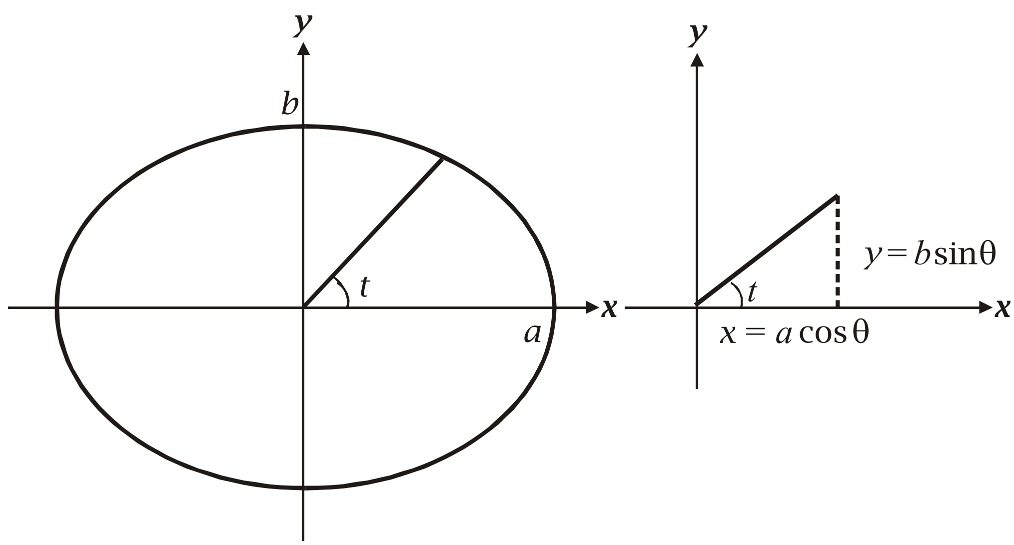
and since $y = b\sin t$ then $\sin t = \frac{y}{b}$

Hence, since

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Hence, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Notice here that when

$t = 0$ then $a\cos t = a,\ b\sin t = 0$

$t = \pi/2$ then $a\cos t = 0,\ b\sin t = b$

so the curve cuts the *x*-axis at *a* and the *y*-axis at *b*.

Here the *x*-axis and *y*-axis are axes of symmetry. The longer axis of symmetry is called the *major axis* and the shorter axis of symmetry is called the *minor axis*.

When $a = b$ the equation of the ellipse reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

or

$$x^2 + y^2 = a^2$$

which is the equation of the circle. So a circle is a special form of ellipse.

Example (7)

Show that the equation for the tangent to the ellipse in parametric form is

$$\frac{y}{b}\sin t + \frac{x}{a}\cos t = 1$$

Solution

By differentiating the parametric form of the ellipse we obtain the tangent to the ellipse

$$x = a\cos t$$
$$y = b\sin t$$
$$\therefore \frac{dx}{dt} = -a\sin t$$
$$\frac{dy}{dt} = b\cos t$$
$$\therefore \frac{dy}{dx} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t$$

By substituting

$$x = a\cos t$$
$$y = b\sin t$$
$$m = \frac{dy}{dx} = -\frac{b}{a}\cot t$$

into $y = mx + c$

we obtain the equation of the tangent to an ellipse. Substitution gives

$$b\sin t = -\frac{b}{a}\cot t\left(a\cos t\right) + c$$
$$b\sin t = -\frac{b}{a}\frac{\cos t}{\sin t}\left(a\cos t\right) + c$$
$$c = b\sin t + \frac{b\cos^2 t}{\sin t} = \frac{b(\sin^2 t + \cos^2 t)}{\sin t} = \frac{b}{\sin t}$$

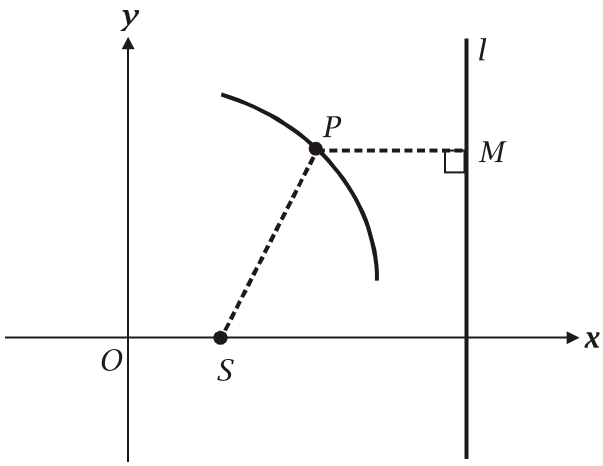
Hence

$$y = -\frac{b}{a}\cot t x + \frac{b}{\sin t}$$

Dividing by b and multiplying by $\sin t$ gives

$$\frac{y}{b}\sin t = -\frac{x}{a}\cot t.\sin t + 1$$
$$\frac{y}{b}\sin t + \frac{x}{a}\cos t = 1$$

The ellipse can also be defined as a locus of a point. This is a point that maintains a distance from both a point and a line.



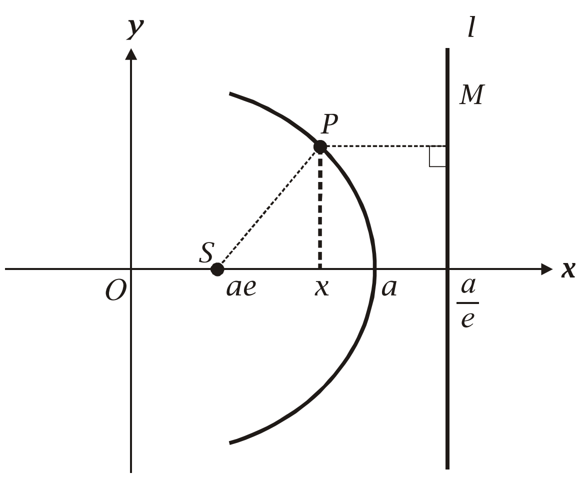
The fixed line is called the *directrix* and the point is called the *focus*. The locus is maintained by the rule that the ratio of PS = the distance of the locus to the focus and PM = the perpendicular distance of the locus to the line is always constant. That is

$\frac{PS}{PM} = e$ where e is a constant.

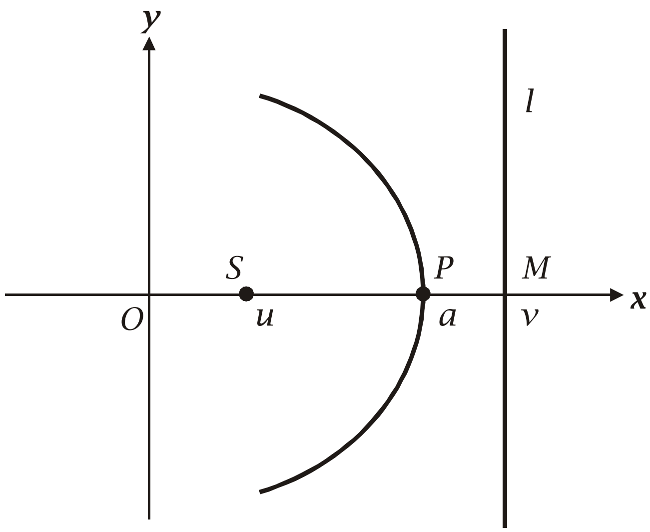
In order to represent an ellipse, the constant e must be less than 1

$0 \le e < 1$

The constant, e, is called the *eccentricity*. The equation $\frac{PS}{PM} = e$ is a third way of defining the ellipse algebraically. If the ellipse cuts the x-axis at a, then we can show that the focus is at the point ae and the directrix passes through the point $\frac{a}{e}$ on the x-axis.



To show this, suppose P lies on the x-axis, and the distances $OS = u$, $OM = v$.



We have

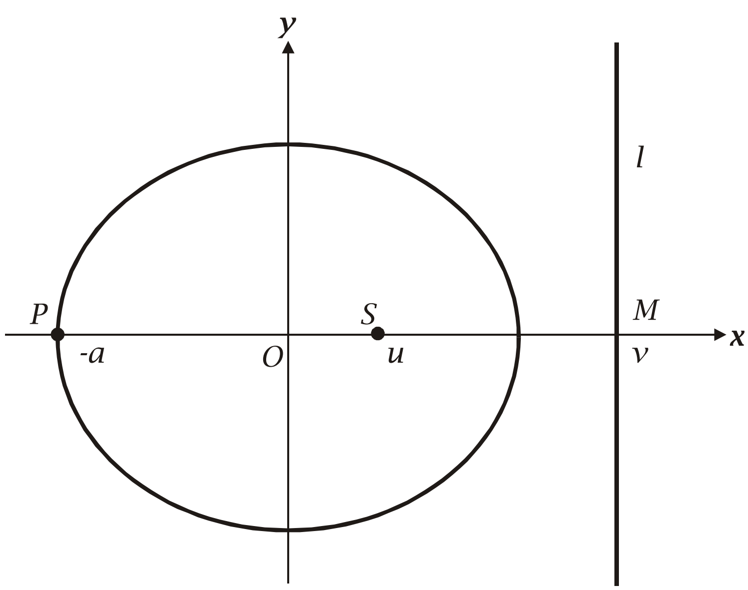
$$\frac{PS}{PM} = e$$

Hence

$$\frac{a-u}{v-a} = e$$

$$a - u = e(v - a) \quad (1)$$

The ellipse also passes through the point $-a$ on the negative x-axis



Hence

$$\frac{PS}{PM} = e$$

$$\frac{a+u}{v+a} = e$$

$$a + u = e(v + a) \quad (2)$$

On adding the two equations

$$2a = e(v - a + v + a)$$

$$2a = 2ev$$

$$v = \frac{a}{e}$$

On subtracting the two equations

$$-2u = e(v - a - v - a)$$

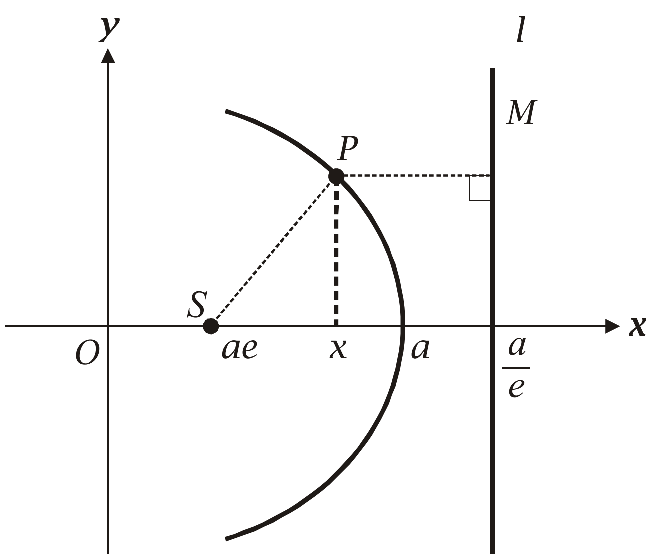
$$2u = 2ea$$

$$u = ea$$

This demonstrates that the focus lies at ea and the directrix lies at $\frac{a}{e}$ on the x-axis. These properties enable us to derive a fourth equation representing the ellipse

$$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

To show this



$$\frac{PS}{PM} = e$$

Hence

$$\frac{PS^2}{PM^2} = e^2$$

$$PS^2 = e^2 PM^2$$

But, $PS^2 = (x - ae)^2 + y^2$ and $PM^2 = \left(\frac{a}{e} - x\right)^2$

Whence

$$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$x^2 - 2ae + (ae)^2 + y^2 = a^2 - 2ae + e^2x^2$$
$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

If we divide both sides of this equation by a^2 and $(1-e)^2$ we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Which, if we compare with the Cartesian equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ shows us that

$$b^2 = a^2(1 - e^2)$$

Example (8)

An ellipse has Cartesian equation

$$\frac{x^2}{5} + \frac{y^2}{4} = 1$$

Find its eccentricity, the coordinates of its foci and the equations of its two directrices.

Solution

The equation of the ellipse in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore a^2 = 5,\ b^2 = 4$$
$$\therefore a = \sqrt{5},\ b = 2$$

Substitution into the relationship

$$b^2 = a^2(1 - e^2)$$

gives

$$4 = 5(1 - e^2) \quad \Rightarrow \quad 4 = 5 - 5e^2 \quad \Rightarrow \quad 5e^2 = 1 \quad \Rightarrow \quad e = \sqrt{1/5}$$

The foci are at $\pm ae$; hence, the foci are at $\pm\sqrt{5}.\sqrt{1/5} = \pm 1$.

The equations of the directrix are $x = \pm\frac{a}{e}$

Hence, the equations of the directrix are $x = \pm\frac{1}{\sqrt{1/5}} = \pm\sqrt{5}$

The hyperbola

The hyperbola has parametric equations

$x = a\sec t$

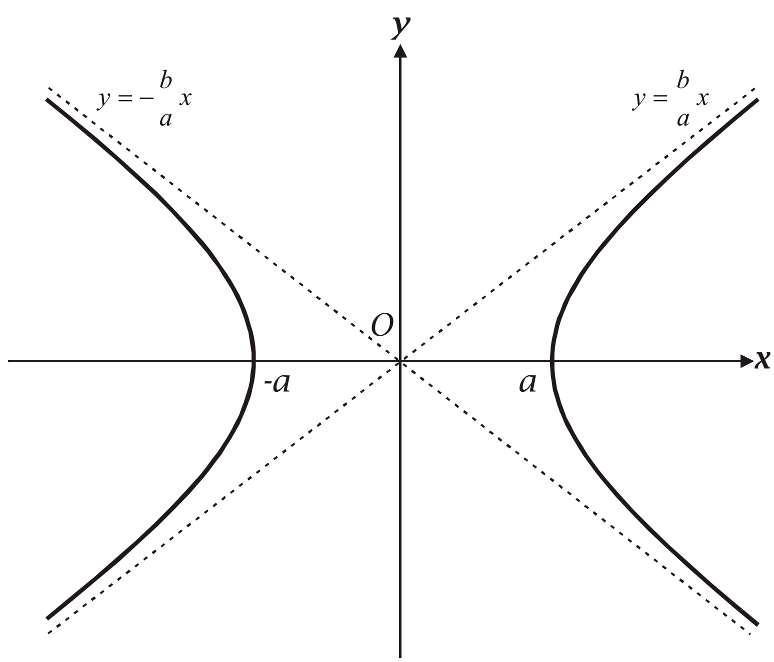
$y = b\tan t$

where $0 \le t < 2\pi$

The Cartesian equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

This can be derived from the parametric equations as follows. Since $x = a\sec t$, $\sec t = \frac{x}{a}$ and since $y = b\tan t$, $\tan t = \frac{y}{b}$ substitution into the trigonometric identity $\sec^2 x - \tan^2 x = 1$ gives

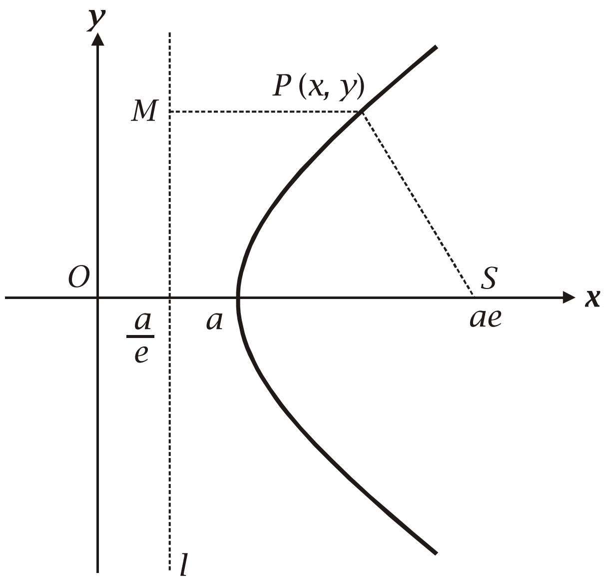
$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$



The *x*-axis here is called the *transverse axis*, and the *y*-axis is the *conjugate axis*. The central point through which these axes of symmetry pass is called the centre. The asymptotes are the lines to which the curve gets closer and closer without actually touching. In this standard representation of a hyperbola, the asymptotes are the lines

$y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$

The focus/directrix property also applies to hyperbolas. This is illustrated by the following diagram.



The ratio of the distances

$$\frac{PS}{PM} = e \quad e > 1$$

is constant. That is, a hyperbola can be defined to be the locus of a point such that the distance of that point from a fixed point, *S*, called the *focus*, and a fixed line, *l*, called the *directrix*, is always a constant number¸ *e*, where $e > 1$. As for the ellipse we can show that the directrix is the line $y = \frac{a}{e}$, and the focus lies at *ae* on the *x*-axis. The hyperbola has two branches, and the second directrix lies at $y = -\frac{a}{e}$ and the second focus lies at –*ae*. The equation for the hyperbola can be made to take the form

$$\frac{x^2}{a^2} - \frac{y^2}{a^2\left(e^2 - 1\right)} = 1$$

and, by comparison with the standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we have

$$b^2 = a^2\left(e^2 - 1\right)$$

Example (9)

A hyperbola has Cartesian equation

$$\frac{x^2}{144} - \frac{y^2}{25} = 1$$

Find its eccentricity, the coordinates of the foci and the equations of its directrices.

Solution

The standard form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Comparison with this form gives

$a = 12,\ b = 5$

Substitution into the relationship

$b^2 = a^2\left(e^2 - 1\right)$

gives

$25 = 144\left(e^2 - 1\right)$

$144e^2 - 144 = 25$

$144e^2 = 169$

$e = 13/12$

Foci are at $(\pm ae, 0)$, hence at $\pm\left(13/12, 0\right)$

Directrices are at $x = \pm\frac{a}{e}$, hence at $x = \pm\frac{12}{13/12} = \pm\frac{144}{13}$

The rectangular hyperbola

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ gives the general form of the hyperbola. When $a = b$, then the hyperbola takes a special form called the rectangular hyperbola with equation

$x^2 - y^2 = a^2$

In this case since

$b^2 = a^2\left(e^2 - 1\right)$ and $b = a$, then

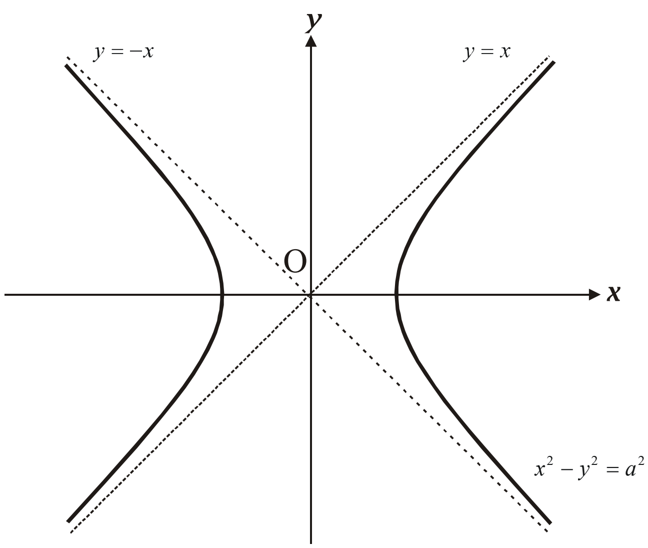
$a^2 = a^2\left(e^2 - 1\right)$

$\therefore e^2 - 1 = 1$

$e^2 = 2$

$e = \sqrt{2}$

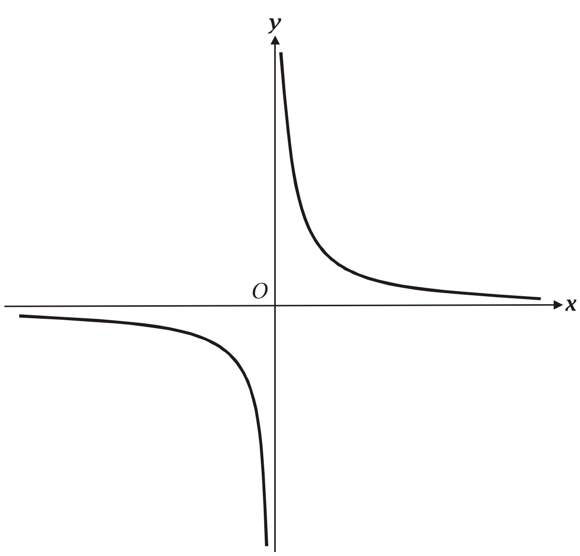
So the eccentricity of a rectangular hyperbola is $\sqrt{2}$. The asymptotes of a rectangular hyperbola in standard form are $y = x$ and $y = -x$.



Rotation of the rectangular hyperbola about the origin by 45° gives the curve

$xy = c^2$ or $y = \frac{c^2}{x}$

in which the asymptotes are the coordinate axes



Example (10)

Show that the equation of the tangent to a rectangular hyperbola in standard position with parametric equations

$x = a\sec t$

$y = a\tan t$

is

$x\sec t - y\tan t = a$

Solution

The parametric equations of a rectangular hyperbola are

$x = a\sec t$

$y = a\tan t$

Therefore, on differentiating

$$\frac{dx}{dt} = a\sec t\tan t$$

$$\frac{dy}{dt} = a\sec^2 t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sec^2 t}{a\sec t\tan t} = \frac{\sec t}{\tan t} = \frac{1/\cos t}{\sin t/\cos t} = \frac{1}{\sin t}$$

On substituting the gradient into the equation of the straight line

$y = mx + c$

we obtain

$$y = \frac{1}{\sin t}x + c$$

Substituting the coordinates of a point on the curve, $x = a\sec t,\ y = a\tan t$

$$a\tan t = \frac{1}{\sin t}a\sec t + c$$

$$a\frac{\sin t}{\cos t} = \frac{1}{\sin t}a\frac{1}{\cos t} + c$$

$$c = a\left(\frac{\sin t}{\cos t} - \frac{1}{\sin t\cos t}\right)$$

$$c = a\left(\frac{\sin^2 t - 1}{\sin t\cos t}\right) = a\left(\frac{-\cos^2 t}{\sin t\cot}\right) = -a\frac{\cos t}{\sin t}$$

Then the equation of the tangent is

$$y = \frac{1}{\sin t}x - a\frac{\cos t}{\sin t}$$

on multiplying through by $\frac{\sin t}{\cos t}$

$$y\frac{\sin t}{\cos t} = \frac{1}{\cos t}x - a$$

$$y\tan t = x\sec t - a$$

Conics in standard position

Conic	**Eccentricity**	**Focus**	**Directrix**	**Cartesian equation**	**Parametric equation**
Circle	---	---	---	$x^2 + y^2 = r^2$	$x = a\cos t$ $y = a\sin t$ where $0 \le t < 2\pi$
Ellipse	$0 < e < 1$	$(\pm ae, 0)$	$x = \pm\frac{a}{e}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = a^2(1 - e^2)$	$x = a\cos t$ $y = b\sin t$ where $0 \le t < 2\pi$
Parabola	$e = 1$	$(a, 0)$	$x = -a$	$y^2 = 4ax$	$x = at^2$ $y = 2at$ where a is a positive constant.
Hyperbola	$e > 1$	$(\pm ae, 0)$	$x = \pm\frac{a}{e}$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$	$x = a\sec t$ $y = b\tan t$ where $0 \le t < 2\pi$
Rectangular Hyperbola	$e = \sqrt{2}$	$(\pm a\sqrt{2}, 0)$	$x = \pm\frac{a}{\sqrt{2}}$	$x^2 - y^2 = a^2$	$x = a\sec t$ $y = a\tan t$ where $0 \le t < 2\pi$