Conservation of Angular Momentum

When two objects are involved in a head-on collision we believe that their total linear momentum is conserved – this is the substance of Newton's famous second law. The momentum of the combined objects before the collision is equal to the moment of the combined objects after the collision.

Forces not only propel objects in straight lines; they also cause them to rotate. Rotating objects have angular momentum. So we would expect the angular momentum of objects involved in a collision also to be conserved – that is, the sum of the angular momentums of the particles before a collision is equal to the sum of the angular momentums of those particles after collision.

Thus we seek an analogy to the familiar result that in any closed system linear momentum is conserved for angular momentum.

That analog is the conservation of angular momentum: the angular momentum of any closed system that is rotating is conserved – that is, does not change.

By analogy with linear momentum, angular momentum, is defined to be:

angular momentum = moment of inertia × angular velocity

$$J = I\dot{\theta}$$

By analogy with conservation of linear momentum, conservation of angular momentum entails that, for any closed system, the sum of all the angular momentums of the constituent parts is always constant.

$$\frac{dJ}{dt} = 0 \quad \frac{d}{dt}\left(I\theta\right) = 0$$

Angular momentum is also called moment of momentum.

If a system is subject to an external torque (also called moment or couple) then the angular momentum of such as system will not be conserved. Such a system is said to be open. In an open system angular momentum is not conserved

Firstly, we illustrate the application of conservation of momentum; then we demonstrate the consistency of the definition of angular momentum and prove the theorem that in a closed system angular momentum is always conserved.

Example (1)

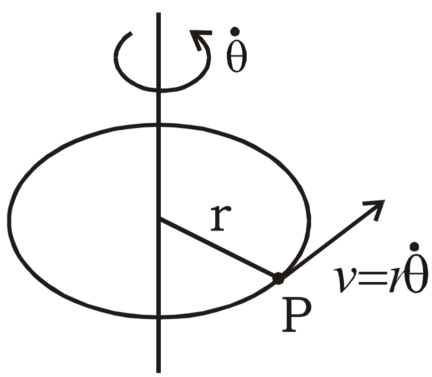
An ice-skater rotates about a fixed vertical axis with moment of inertia 6.0 Kg m^2 when her arms are extended. She draws her arms to her side, and her moment of inertia changes to 2.4 Kg m^2. Given that her initial angular velocity is 7.2 $rads^{-1}$ find her final angular velocity once she has brought her arms down by her side.

Conservation of angular momentum gives

$$I_1\omega_1 = I_2\omega_2$$
$$\therefore 6.0 \times 7.2 = 2.4 \cdot \omega_2$$
$$\omega_2 = \frac{6.0 \cdot 7.2}{2.4} = 18 \text{ rads}^{-1}$$

We now demonstrate the consistency of the definition of angular momentum with the definition of linear momentum.

Let *P* be a particle of mass *M* rotating about an axis *L*. Let *r* be the perpendicular distance of *P* from *L*.



As usual the tangental velocity is

$$v = r\dot{\theta}$$

Then the tangental momentum is

$$mr\dot{\theta}$$

Note that since P is in orbit there is no radial component of the linear momentum, so the tangental linear momentum is all the momentum there is.

Now the moment of a force – its torque – is defined to be:

moment = force × perpendicular distance

$C = Fr.$

By analogy, the moment of the linear momentum (called moment of momentum or angular momentum) is

moment of momentum = linear momentum x perpendicular distance

$$\begin{aligned} J &= mv \times r \\ &= mr\dot{\theta} \times r \\ &= mr^2\dot{\theta} \end{aligned}$$

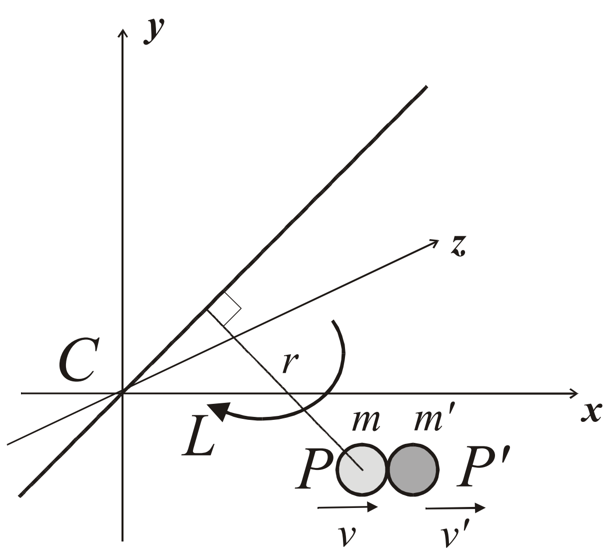
$$\therefore J = I\dot{\theta}$$

since $I = mr^2$

Proof of the result

We now proceed to prove the main result – that in any collision, or closed system, angular momentum is conserved.

Consider the collision of two particles P and P' of mass M and M' and velocities V and V' respectively. Let the collision be modelled by the collision of particles of no size at a distance r from a point O.



Let $I = mr^2$ be the moment of inertia of P about any axis of rotation L through O.

Let $I' = m'r^2$ be the equivalent moment of inertia of P'.

As a result of the collision P imparts an impulse, that is a change of momentum to P'. By the conservation of linear momentum P' imparts a change in momentum to P that is equal and opposite to this impulse. Thus

$$\Delta mv = \Delta m'v'$$
$$\therefore \Delta m\dot{\theta}r = \Delta m'\dot{\theta}'r$$

multiplying both sides by r:

$$\Delta m\dot{\theta}r^2 = \Delta m'\dot{\theta}'r^2$$
$$\therefore \left(\Delta mr^2\right)\dot{\theta} = \left(\Delta m'r^2\right)\dot{\theta}'$$
$$\therefore I\dot{\theta} = I'\dot{\theta}'$$

Thus demonstrating conservation of angular momentum.

Rotational kinetic energy

Just as the kinetic energy of an object with linear momentum is

$$E = \tfrac{1}{2}mv^2$$

so the rotational kinetic energy of an object with angular momentum will be

$$E = \tfrac{1}{2}\mathrm{I}\omega^2$$

Where I = moment of inertia
and ω = angular velocity

Also, in any collision, kinetic energy must be conserved. Likewise, if a rotating object slows down, so that it loses rotational kinetic energy, then that energy is converted to other forms of energy, without loss.

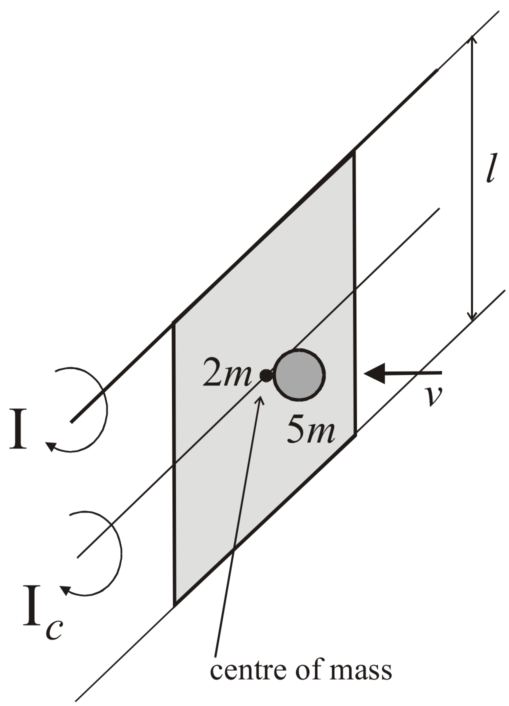
The use of energy in solving problems is illustrated by the next example.

Example

A uniform square lamina of mass $2m$ and side l is hinged along one edge, and thus is free to rotate about a fixed, smooth, horizontal axis which coincides with a side of the lamina. The lamina is hanging in equilibrium when a particle of mass $5m$ moving with speed v in a direction perpendicular to the plane of the lamina strikes it at its centre of mass. The particle sticks to the lamina. Find, in terms of v and a, the angular speed of the lamina immediately after the impact.

Hence show that the lamina will perform complete revolutions if $v^2 > \frac{322}{75} lg$.

Solution



The side of the lamina is l so the distance of the centre of mass from the hinge is

$$d = \frac{l}{2}$$

The moment of inertia of the lamina is given by the parallel axis theorem

$$\mathrm{I} = \mathrm{I}_c + 2md^2$$

Since the moment of inertia of a lamina of side l is

$$I_c = \frac{2m\frac{l^2}{4}}{3}$$

We have

$$I = \frac{2m\frac{l^2}{4}}{3} + 2m\frac{l^2}{4} = \frac{4ml^2}{6} = \frac{2ml^2}{3}$$

The particle of mass $5m$ is travelling with a velocity of v. At impact its distance from the hinge is $\frac{l}{2}$, so its angular momentum at that instant, using the equation

moment of momentum = linear momentum x perpendicular distance

is

$$J = 5mv\left(\frac{l}{2}\right)$$

This is imparted to the angular momentum of the combined lamina and particle after the collision. By the principle of conservation of angular momentum we obtain

$$5m \times \left(\frac{l}{2}\right) \times v = 5m\left(\frac{l^2}{4}\right)\omega + \frac{2ml^2}{3}\omega$$

$$30mlv = 15ml^2\omega + 8ml^2\omega$$

$$30v = 23l\omega$$

$$\omega = \frac{30v}{23l}$$

To perform complete revolutions the lamina has to reach the uppermost position.

At impact the moment of inertia of the particle is

$$I = \frac{1}{2} \times 5m \times \left(\frac{l}{2}\right)^2 = \frac{5}{8}ml^2$$

By the conservation of the energy we find that the rotational kinetic energy of the combined lamina and particle at impact must be greater than the gravitational potential energy of the centre of mass at the uppermost position. Thus

$$\frac{1}{2}\mathrm{I}\omega^2 + \frac{5}{8}ml^2 \times \omega^2 > \left(2ml + 5ml\right)g$$

$$\frac{ml^2}{3}\omega^2 + \frac{5ml^2}{8}\omega^2 > 7mlg$$

$$\frac{23ml^2}{24}\omega^2 > 7mlg$$

$$\omega^2 > \frac{168g}{23l}$$

$$\frac{900v^2}{529l^2} > \frac{168g}{23l}$$

$$v^2 > \frac{3864gl}{900} = \frac{322gl}{75}$$