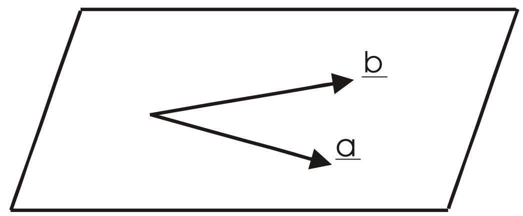
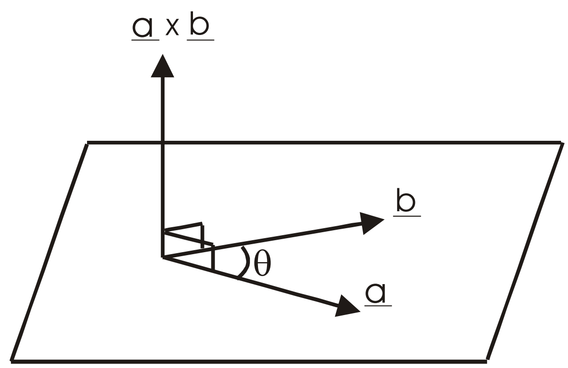
Cross product of two vectors

In two dimensions any two vectors define a plane.



We wish to find a vector that is perpendicular to the two vectors **a** and **b**. This vector is the cross product of **a** and **b** and is denoted **a** × **b**



Let θ be the angle between the vectors **a** and **b**. Note that θ need not be a right angled triangle. Then **a** x **b** is defined to be

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \left\{ |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \sin\theta \right\} \hat{\underline{\mathbf{c}}}$$

where $\hat{\underline{\mathbf{c}}}$ is a unit vector perpendicular to both **a** and **b**.

We find the cross product of two vectors in three dimensions by a technique of expanding a determinant as follows.

Suppose

$\underline{\mathbf{a}} = a_1 \underline{\mathbf{i}} + a_2 \underline{\mathbf{j}} + a_3 \underline{\mathbf{k}}$

and

$\underline{\mathbf{b}} = b_1 \underline{\mathbf{i}} + b_2 \underline{\mathbf{j}} + b_3 \underline{\mathbf{k}}$

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Then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \underline{\mathbf{i}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \underline{\mathbf{j}} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \underline{\mathbf{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \underline{\mathbf{i}}\left(a_2b_3 - a_3b_2\right) + \underline{\mathbf{j}}\left(a_3b_1 - a_1b_3\right) + \underline{\mathbf{k}}\left(a_1b_2 - a_2b_1\right)$$

Example

Find $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ when

$\underline{\mathbf{a}} = 2\underline{\mathbf{i}} - 3\underline{\mathbf{j}} + 4\underline{\mathbf{k}}$ and $\underline{\mathbf{b}} = -\underline{\mathbf{i}} - \underline{\mathbf{j}} - 3\underline{\mathbf{k}}$

Solution

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 2 & -3 & 4 \\ -1 & -1 & -3 \end{vmatrix}$$

$$= \underline{\mathbf{i}} \begin{vmatrix} -3 & 4 \\ -1 & -3 \end{vmatrix} + \underline{\mathbf{j}} \begin{vmatrix} 4 & 2 \\ -3 & -1 \end{vmatrix} + \underline{\mathbf{k}} \begin{vmatrix} 2 & -3 \\ -1 & -1 \end{vmatrix}$$

$$= \underline{\mathbf{i}}\left(9+4\right) + \underline{\mathbf{j}}\left(-4+6\right) + \underline{\mathbf{k}}\left(-2-3\right)$$

$$= 13\underline{\mathbf{i}} + \underline{2\mathbf{j}} - 5\underline{\mathbf{k}}$$

To confirm that this is the case we will find the dot (scalar) products of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ with $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ separately:

$$\left(\underline{\mathbf{a}} \times \underline{\mathbf{b}}\right) \cdot \underline{\mathbf{a}} = \left(13, 2, -5\right) \cdot \left(2, -3, 4\right) = 26 - 6 - 20 = 0$$

Hence, $\left(\underline{\mathbf{a}} \times \underline{\mathbf{b}}\right)$ and $\underline{\mathbf{a}}$ are perpendicular.

Similarly

$$\left(\underline{\mathbf{a}} \times \underline{\mathbf{b}}\right) \cdot \underline{\mathbf{b}} = (13, 2, -5) \cdot (-1, -1, -3) = 13 - 2 + 15 = 0$$

Hence, $\left(\underline{\mathbf{a}} \times \underline{\mathbf{b}}\right)$ and $\underline{\mathbf{b}}$ are perpendicular.

When the cross product has not been studied a perpendicular vector to two vectors must be found by using the properties of the dot product and solving a system of simultaneous equations.

<u>Example</u>

For example, find a vector perpendicular to $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ when

$$\underline{\mathbf{a}} = 2\underline{\mathbf{i}} - 3\underline{\mathbf{j}} + 4\underline{\mathbf{k}}$$
$$\underline{\mathbf{b}} = -\underline{\mathbf{i}} - \underline{\mathbf{j}} - 3\underline{\mathbf{k}}$$

Solution

Let the vector $\underline{\mathbf{r}} = a\underline{\mathbf{i}} + b\underline{\mathbf{j}} + c\underline{\mathbf{k}}$ be perpendicular to $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

Then $\underline{\mathbf{r}} \cdot \underline{\mathbf{a}} = 0$ and $\underline{\mathbf{r}} \cdot \underline{\mathbf{b}} = 0$

Hence

$(a, b, c).(2, -3, 4) = 0$

$2a - 3b + 4c = 0$ (1)

and

$(a, b, c).(-1, -1, -3) = 0$

$-a - b - 3c = 0$ (2)

On multiplying $(2) \times 2$

$-2a - 2b - 6c = 0$

Then $(1) + (3)$ gives:-

$-5b - 2c = 0$

$2c = -5b$

If $c = 5$ $b = -2$ and $a = -13$

Hence $\underline{\mathbf{r}} = -13\underline{\mathbf{i}} - 2\underline{\mathbf{j}} + 5\underline{\mathbf{k}}$ is perpendicular to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

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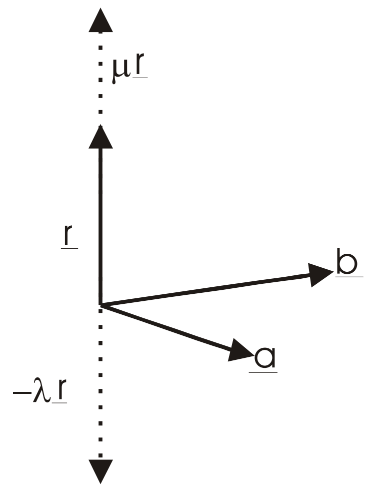
The technique using determinants is easier if you know it!

The perpendicular vector to **a** and **b** is not unique. Any scalar multiple of a vector perpendicular to **a** and **b** is also perpendicular to **a** and **b**.



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