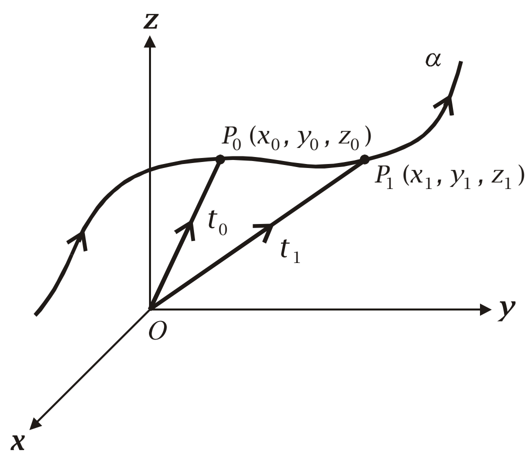
Curves in Euclidean Space

Curves in Euclidean space

Here we are considering curves embedded in three-dimensional Euclidean space, $\mathbb{E}^3$. Our aim here is to find an appropriate mathematical description of such a curve.



As the diagram indicates one way to proceed is by giving the Euclidean coordinates of an arbitrary point *P* on a curve *α*.

$P = (x, y, z)$

To define a curve we must allow *P* to move along the curve, so we shall specify *P* as a point moving along the curve with respect to a parameter *t*. This makes the coordinates of *P* into coordinate functions of the parameter *t*.

$P(t) = (x(t), y(t), z(t))$

Different values of *t* shall specify different points on the curve. Thus at time $t = 0$ the point on the curve shall be

$P_0 = P(0) = (x(0), y(0), z(0)) = (x_0, y_0, z_0)$.

At $t = 1$ we shall have

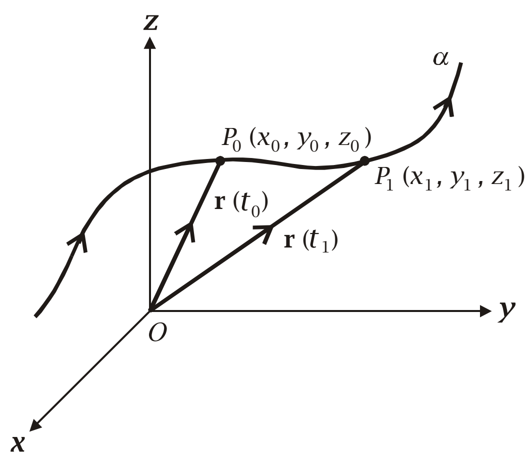
$P_1 = P(1) = (x(1), y(1), z(1)) = (x_1, y_1, z_1)$

and so forth.

The parameter *t* may be given a physical interpretation. One obvious interpretation is to use *t* to represent time. In that case the distance between two points P_0 and P_1 along a curve represents the distance travelled in the given time by a particle moving along that curve. The first derivative gives the velocity of a particle travelling along the curve and the second derivative gives its acceleration. But the parameter t need not represent time, it may be completely arbitrary. Also, there is the unit speed parametrization of a curve – this is the parameter that travels along the curve at unit speed $\left(1\ \text{ms}^{-1}\right)$.

Another advantage of this approach is that it allows the curve to be traversed in two different directions. If we substitute $-t$ for t in $P(t)=\left(x(t),y(t),z(t)\right)$ we get the same curve travelled in the opposite direction.

Clearly $\overrightarrow{OP}$ is a vector from O to the point P on the curve. Hence, this approach turns a curve in three-dimensional Euclidean space into a continuous vector function.



So

$P(t)=\mathbf{r}(t)=\left(x(t),y(t),z(t)\right)$

Example (1)

A curve in Euclidean 3-space is given by

$\mathbf{r}(t)=(x,y,z)=(a\cos t,a\sin t,1)$

By eliminating t from $x(t)$ and $y(t)$ find the Cartesian equation linking x and y. Hence describe this curve geometrically.

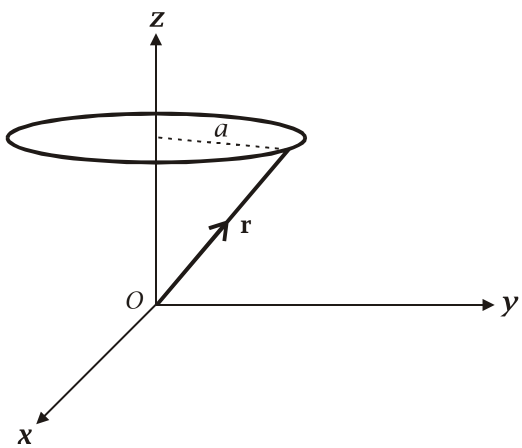
Solution

$x^2+y^2=a^2\cos t+a^2\sin t=a^2$

$x^2+y^2=a^2$ is the Cartesian equation of a circle centre the origin and of radius a. We have

$$\begin{cases} x^2 + y^2 = a^2 \\ z = 1 \end{cases}$$

Hence this traces out the circle centred on the z-axis of radius a in the plane $z = 1$.



It is not possible to eliminate t from $(x(t), y(t), z(t))$ to get a Cartesian equation for a **curve** as a relationship between x, y and z. If we can by algebraic manipulations obtain a form $z = f(x, y)$ this represents a Cartesian equation for a two-dimensional **surface** embedded in 3-dimensional Euclidean space. In the Cartesian equation $z = f(x, y)$ we see that z is a function of two variables x and y and hence describes a surface and not a curve.

Example (2)

By letting $z = 0,\ z = \pm 1,\ z = \pm 2,\ z = \pm 3$

sketch the surface $z = \pm\sqrt{x^2 + y^2}$.

Solution

We get a series of *contour curves* as relationships between x and y for these different values of z.

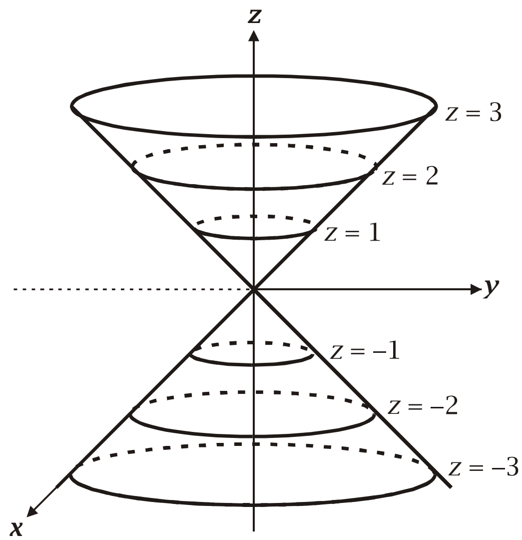
$z = 0 \quad \Rightarrow \quad x^2 + y^2 = 0 \quad \Rightarrow \quad x = y = 0$

$z = \pm 1 \quad \Rightarrow \quad x^2 + y^2 = 1$

$z = \pm 2 \quad \Rightarrow \quad x^2 + y^2 = 2^2$

$z = \pm 3 \quad \Rightarrow \quad x^2 + y^2 = 3^2$

The surface $z = \pm\sqrt{x^2 + y^2}$ is a cone generated by rotating the line $z = x$ about the z-axis.



The vector $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ traces out the straight line through A and B where **a**, **b** are the position vectors of A, B respectively.

Example (3)
Find the parametric form of the straight line between the points $A(1,-1,0)$ and $B(0,1,-1)$ when $t = 0$ s and $t = 1$ s respectively. If t represents the time of a particle P on this curve, find the speed of P as it moves along this curve.

Solution

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 0, \\ 1 \\ -1 \end{pmatrix} \mathbf{b} - \mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = (1-t, 2t-1, -t)$$

The distance $|AB| = |\mathbf{b} - \mathbf{a}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$, which the particle traverses in 1 second. So the speed is: $\text{speed} = \sqrt{6}\ \text{ms}^{-1}$

Clearly the vector $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ is not the only parametrization of the straight line between two points A and B where **a**, **b** are the position vectors of A, B respectively.

Curves are continuous vector functions of a single parameter

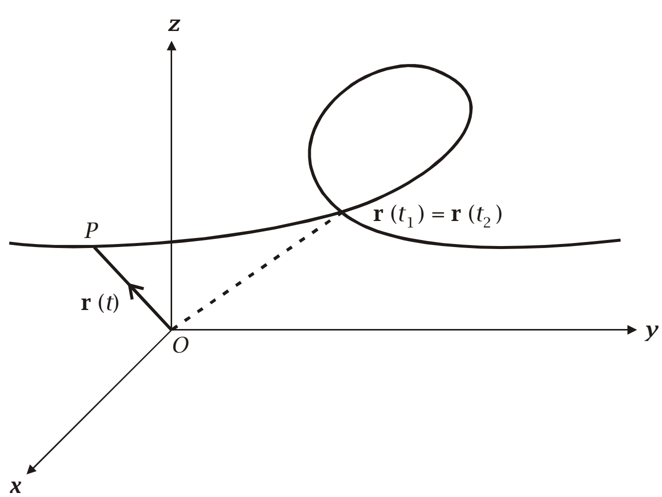
A vector function is a mapping

$$\begin{cases} \mathbb{R} \to \mathbb{E}^3 \\ t \mapsto \mathbf{r}(t) = (f_1(t), f_2(t), f_3(t)) \end{cases}$$

If t is a continuous variable and f_1, f_2, f_3 are continuous then $\mathbf{r}$ is a continuous vector function. Curves in Euclidean space are continuous vector functions. The function $\mathbf{r}(t) = (t, t^2, t^3)$ is an example of a continuous vector function. The function $\mathbf{r}(t) = \left(\frac{1}{t}, 1, 1\right)$ is not continuous because there is a singularity at $t = 0$ around the neighbourhood of which $\frac{1}{t}$ varies from $+\infty$ to $-\infty$. In the expression

$$\begin{cases} \mathbb{R} \to \mathbb{E}^3 \\ t \mapsto \mathbf{r}(t) = (f_1(t), f_2(t), f_3(t)) \end{cases}$$

the domain of the vector function is the real line $\mathbb{R}$. It is not necessary for the domain to be the entire set $\mathbb{R}$; it may be any *open* subset of $\mathbb{R}$. Denote by I an open interval in the real line $\mathbb{R}$; that is $I = (a, b)$ where a, b are real numbers. Hence the parameter t is such that $a < t < b$. In order for the vector function to be continuous, it is essential that the set be *open*. That means that the end-points a and b are not included in the set. Thus if we let $\overrightarrow{OP}$ be a vector where O is the origin and P is the point given by the continuous vector function $\mathbf{r}(t) = (f_1(t), f_2(t), f_3(t))$, then as t varies $\overrightarrow{OP}$ describes a continuous curve in Euclidean space. The vector function $\overrightarrow{OP} = \mathbf{r}(t)$ is called the parametric equation of the curve. It is not necessary that $\mathbf{r}(t)$ is a bijection (one-one) as a curve may intersect itself; for example, if t is time then at two different times t_1, t_2 it is possible that $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ but not $t_1 = t_2$.

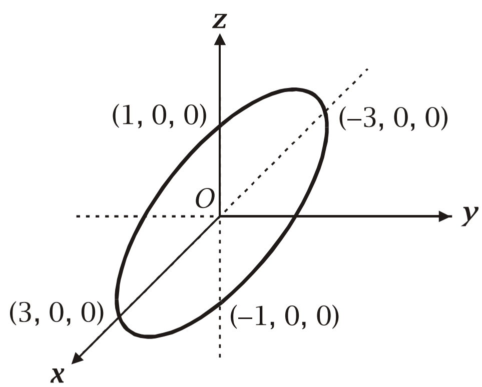


Example (4)

Sketch $\mathbf{r} = (3\cos \pi t, 0, \sin \pi t)$

Solution

It is an ellipse in the xz-plane $y = 0$ passing through $(\pm 3,0,0)$ and $(0,0,\pm 1)$.

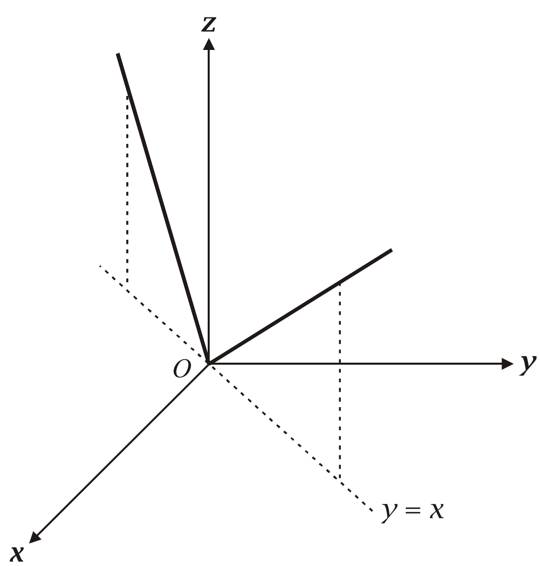


Example (5)

Sketch $\mathbf{r} = (t, t, |t|)$

Solution

It is a straight-line wedge in the plane spanned by $x = y$ and the z-axis.



Example (6)

Sketch $\mathbf{r} = (t, t^2, t^3 - t)$

Solution

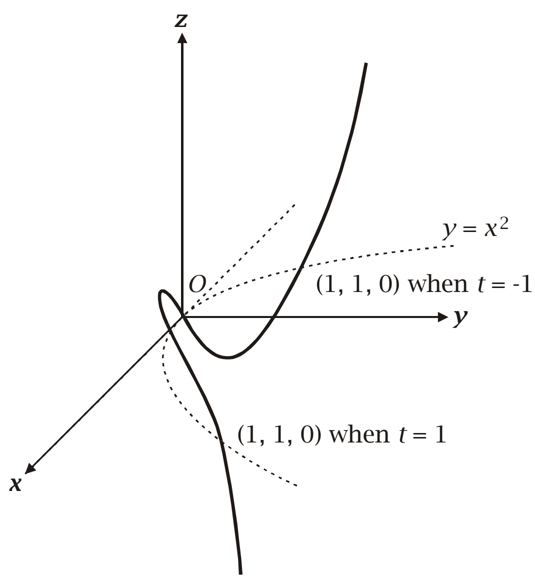
Factorising this gives

$\mathbf{r} = \left(t, t^2, t^3 - t\right) = \left(t, t^2, t\left(t^2 - 1\right)\right)$

That is

$(x, y, z) = \left(t, t^2, t\left(t^2 - 1\right)\right)$

If it were that $z = 0$ this would describe a parabola $y = x^2$ in the *xy*-plane. The value of *z* represents the height above or below this parabola $z = x\left(x^2 - 1\right)$. This is a cubic function with roots at 0, 1 and –1. The curve lies above and below the parabola $y = x^2$ in the *xy*-plane.



Differentiation of continuous vector functions

<u>First Derivative</u>

Let $\mathbf{r}(t) = \left(f_1(t), f_2(t), f_3(t)\right)$ be a continuous vector function of *t*, then

$$\frac{d\mathbf{r}}{dt} = \left(\frac{df_1}{dt}, \frac{df_2}{dt}, \frac{df_3}{dt}\right)$$

is the first derivative of $\mathbf{r}(t)$. This first derivative is also a vector. To show this we must show that $\left(\frac{df_1}{dt}, \frac{df_2}{dt}, \frac{df_3}{dt}\right)$ is (a) a triad of scalars, (b) invariant under translation, and (c) satisfies the transformation law $\frac{d}{dt}\mathbf{r}' = \left(a_{ij}\right)\frac{d}{dt}\mathbf{r}$ where the a_{ij} are the direction cosines of $Ox'y'z'$ relative to $Oxyz$ and $\mathbf{r}' = \left(a_{ij}\right)\mathbf{r}$.

Proof

(a) This is self-evident.

(b) Let $\mathbf{F}(t)$ be translated by $\mathbf{a}$. Thus $\mathbf{r}' = \mathbf{a} + \mathbf{F}(t)$. Then

$$\begin{aligned}\frac{d\mathbf{r}'}{dt} &= \frac{d}{dt}\left(\mathbf{a} + \mathbf{F}(t)\right) \\ &= \frac{d}{dt}\mathbf{a} + \frac{d\mathbf{F}}{dt} && [\text{since } \mathbf{a} \text{ is a constant}] \\ &= \frac{d\mathbf{F}}{dt} && [\text{also since } \mathbf{a} \text{ is a constant}] \\ &= \left(\frac{df_1}{dt}, \frac{df_2}{dt}, \frac{df_3}{dt}\right)\end{aligned}$$

(c) Let $\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be the *transformation matrix* of the rotation of axes. Let $\mathbf{r}' = (a_{ij})\mathbf{r}$ where $\mathbf{r}(t) = (f_1(t), f_2(t), f_3(t))$ is a vector function of *t*. Then

$$\begin{aligned}\frac{d\mathbf{r}'}{dt} &= \frac{d}{dt}(a_{ij})\mathbf{r} \\ &= (a_{ij})\frac{d\mathbf{r}}{dt} && [\text{Since the } a_{ij} \text{ are constant}]\end{aligned}$$

Example (7)

Verify that the curve

$\mathbf{r}(t) = (x, y, z) = (a\cos t, a\sin t, 1)$ (see example (1))

under the transformation matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which represents the transformation when a set of axes is rotated clockwise (i.e. from the *x*-axis to the *y*-axis) through an angle of $\pi/4$ about the *z*-axis, obeys the transformation law $\frac{d}{dt}\mathbf{r}' = (a_{ij})\frac{d}{dt}\mathbf{r}$.

Solution

$$\mathbf{r}' = (a_{ij})\mathbf{r} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a\cos t \\ a\sin t \\ 1 \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} \cos t + \sin t \\ \sin t - \cos t \\ 1 \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{r}' = \frac{d}{dt}\left\{\frac{a}{\sqrt{2}}(\cos t + \sin t, \sin t - \cos t, 1)\right\} = \frac{a}{\sqrt{2}}(\cos t - \sin t, \cos t + \sin t, 0)$$

$$(a_{ij})\frac{d}{dt}\mathbf{r} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dt}(a\cos t, a\sin t, 1)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} a(-\sin t, \cos t, 0) = \frac{a}{\sqrt{2}}(\cos t - \sin t, \cos t + \sin t, 0) = \frac{d}{dt}\mathbf{r}'$$

Higher derivatives

The second and higher derivatives are defined accordingly. For example, if

$\mathbf{r}(t) = (f_1(t), f_2(t), f_3(t))$ then $\frac{d^2}{dt^2}\mathbf{r} = \frac{d}{dt}\left(\frac{d}{dt}\mathbf{r}\right) = \left(\frac{d^2 f_1}{dt^2}, \frac{d^2 f_2}{dt^2}, \frac{d^2 f_3}{dt^2}\right)$.

Example (8)

Find $\frac{d\mathbf{r}}{dr}$ and $\frac{d^2\mathbf{r}}{dt^2}$ where

(i) $\mathbf{r} = (\cos \pi t, 2\sin \pi t, t)$

(ii) $\mathbf{r} = (t, e^t, e^{-t})$

Solution

(i) $\mathbf{r} = (\cos \pi t, 2\sin \pi t, t)$

$$\frac{d\mathbf{r}}{dt} = (-\pi \sin \pi t, 2\pi \cos \pi t, 1)$$

$$\frac{d^2\mathbf{r}}{dt^2} = (-\pi^2 \cos \pi t, -2\pi^2 \sin \pi t, 0)$$

(ii) $\mathbf{r} = (t, e^t, e^{-t})$

$$\mathbf{r}' = \left(1, e^{t}, -e^{-t}\right)$$
$$\mathbf{r}'' = \left(0, e^{t}, e^{-t}\right)$$

Example (9)

Given $\frac{d\mathbf{r}}{dt} = \left(-e^{-t}\left(\cos t + \sin t\right), e^{-t}\left(\cos t - \sin t\right), 0\right)$ and $\mathbf{r}(0) = (1,0,0)$ find $\mathbf{r}(t)$.

<u>Solution</u>

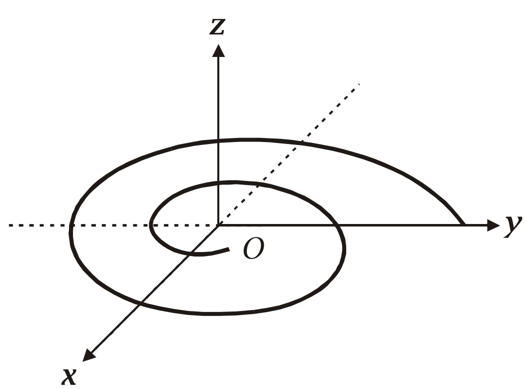
Integrating $\frac{d\mathbf{r}}{dt}$ by inspection or by parts

$$\mathbf{r}(t) = \left(e^{-t}\cos t + A, e^{-t}\sin t + B, C\right)$$

When $t = 0$ we have $C = 0$ and

$$1 + A = 1 \Rightarrow \qquad A = 0$$
$$0 + B = 0 \Rightarrow \qquad B = 0$$

Hence $\mathbf{r}(t) = \left(e^{-t}\cos t, e^{-t}\sin t, 0\right)$. It is a spiral in the *xy*-plane that infinitely spirals in.



Differentiation Rules

$$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(f\mathbf{a}) = \frac{df}{dt}\mathbf{a} + f\frac{d\mathbf{a}}{dt}$$

$$\frac{d}{dt}(\mathbf{a} \bullet \mathbf{b}) = \frac{d\mathbf{a}}{dt} \bullet \mathbf{b} + \mathbf{a} \bullet \frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

These are all proven by writing out the derivatives in component form.

Orthogonality result

Let $\hat{\mathbf{a}}$ be a unit vector. An important result is that the vector $\frac{d\hat{\mathbf{a}}}{dt}$ is perpendicular to $\hat{\mathbf{a}}$, provided that $\frac{d\hat{\mathbf{a}}}{dt} \neq 0$ and $\hat{\mathbf{a}} \neq 0$. To show this we start with the identity

$\hat{\mathbf{a}} \bullet \hat{\mathbf{a}} = |\hat{\mathbf{a}}|^2 = 1$ since $|\hat{\mathbf{a}}| = 1$

Differentiating this gives

$$\frac{d}{dt}(\hat{\mathbf{a}} \bullet \hat{\mathbf{a}}) = 0$$

$$\frac{d\hat{\mathbf{a}}}{dt} \bullet \hat{\mathbf{a}} + \hat{\mathbf{a}} \bullet \frac{d\hat{\mathbf{a}}}{dt} = 0$$

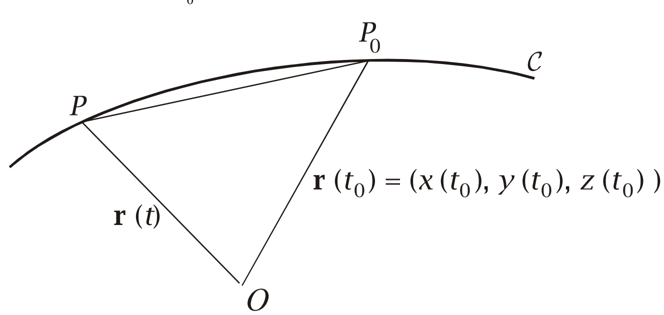
$$2\frac{d\hat{\mathbf{a}}}{dt} \bullet \hat{\mathbf{a}} = 0$$

So, if $\frac{d\hat{\mathbf{a}}}{dt} \neq 0$ this implies that $\frac{d\hat{\mathbf{a}}}{dt}$ and $\hat{\mathbf{a}}$ are vectors that are perpendicular.

Remark: this would apply if **a** were any vector of constant length, so that the length of **a** is not a function of the parameter *t*. However, if the length of **a** varies with the parameter *t* then the result does not follow.

Interpretation of the first derivative as the tangent to a curve

Let $\mathbf{r}(t)$ be a continuous curve, and let PP_0 be a cord joining the points P and P_0 on this curve. The tangent at P_0 is the gradient of this cord in the limit as $P \to P_0$.



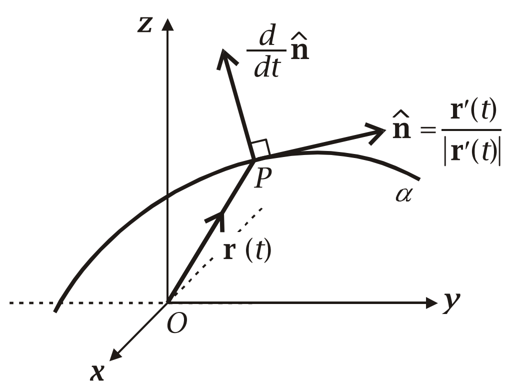
That is the tangent is $\lim_{t \to t_0} \left\{ \frac{PP_0}{t - t_0} \right\}$. However

$$\frac{d\mathbf{r}}{dt} = \lim_{t\to t_0}\left\{\frac{x(t)-x(t_0)}{t-t_0}, \frac{y(t)-y(t_0)}{t-t_0}, \frac{z(t)-z(t_0)}{t-t_0}\right\}$$

$$= \lim_{t\to t_0}\left\{\frac{\mathbf{r}(t)-\mathbf{r}(t_0)}{t-t_0}\right\}$$

$$= \lim_{t\to t_0}\left\{\frac{PP_0}{t-t_0}\right\}$$

So $\frac{d\mathbf{r}}{dt}$ is tangent to the curve, *C*. Furthermore, $\frac{d\mathbf{r}}{dt}$ points in the direction in which *t* increases; i.e. in the direction in which the curve *C* is traced out as *t* increases. Let $\mathbf{n} = \frac{d\mathbf{r}}{dt}$ then $\mathbf{n}$ is tangent to *C* and $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$ is a unit tangent vector to C. We proved above that if $\hat{\mathbf{a}}$ is a unit vector then the vector $\frac{d\hat{\mathbf{a}}}{dt}$ is perpendicular to $\hat{\mathbf{a}}$, provided that $\frac{d\hat{\mathbf{a}}}{dt} \neq 0$. This means that the derivative of $\hat{\mathbf{n}}$, which is $\frac{d}{dt}\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{n}}$.



Example (10)

Let $\mathbf{r} = (\cos t, \sin t, t)$

Show that the vectors $\hat{\mathbf{n}} = \hat{\mathbf{r}}'(t)$ and $\hat{\mathbf{n}}' = \frac{d}{dt}\hat{\mathbf{n}}(t)$ are orthogonal.

<u>Solution</u>

$\mathbf{r} = (\cos t, \sin t, t)$

$\mathbf{r}' = \frac{d\mathbf{r}}{dt} = (-\sin t, \cos t, 1)$

$|\mathbf{r}'| = \sin^2 t + \cos^2 t + 1 = 2$

$\hat{\mathbf{n}} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$

$\hat{\mathbf{n}}' = \frac{d}{dt}\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\cos t, \sin t, 0)$

$\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \cdot \frac{1}{\sqrt{2}}(-\cos t, \sin t, 0) = 0$

Therefore $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ are orthogonal

Example (11)

(a) Find the unit tangent $\hat{\mathbf{T}}$ to the curve $\mathbf{r}(t) = \left(1, t, \frac{t^2}{2}\right)$.

(*b*) Show that $\hat{\mathbf{T}}$ and $\frac{d}{dt}\hat{\mathbf{T}}$ where $\hat{\mathbf{T}}$ is the vector found in part (*a*) are orthogonal.

<u>Solution</u>

(*a*) $\mathbf{T} = \frac{d\mathbf{r}}{dt} = (0, 1, t)$

$|\mathbf{T}| = \sqrt{1 + t^2}$

$\hat{\mathbf{T}} = \frac{1}{\sqrt{1+t^2}}(0, 1, t)$

(*b*) $\hat{\mathbf{T}} = \left(0, \frac{1}{(1+t^2)^{\frac{1}{2}}}, \frac{t}{(1+t^2)^{\frac{1}{2}}}\right)$

$$\frac{d}{dt}\hat{\mathbf{T}} = \left(0, \frac{-t}{(1+t^2)^{\frac{3}{2}}}, \frac{1+t^2-t^2}{(1+t^2)^{\frac{3}{2}}}\right) = \left(0, \frac{-t}{(1+t^2)^{\frac{3}{2}}}, \frac{1}{(1+t^2)^{\frac{3}{2}}}\right)$$

$$\hat{\mathbf{T}} \cdot \frac{d}{dt}\hat{\mathbf{T}} = \left(0, \frac{1}{(1+t^2)^{\frac{1}{2}}}, \frac{t}{(1+t^2)^{\frac{1}{2}}}\right) \cdot \left(0, \frac{-t}{(1+t^2)^{\frac{3}{2}}}, \frac{1}{(1+t^2)^{\frac{3}{2}}}\right)$$

$$= 0 - \frac{t}{(1+t^2)^2} + \frac{t}{(1+t^2)^2} = 0$$

Hence $\hat{\mathbf{T}}$ and $\frac{d}{dt}\hat{\mathbf{T}}$ are orthogonal.

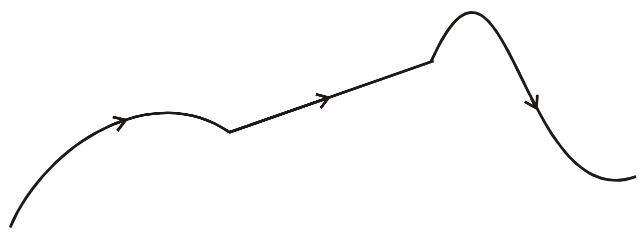
Smooth, piecewise smooth and simple curves

Smooth curve

Intuitively, a smooth curve is one that does not suddenly change direction. Thus a curve $\mathbf{r}(t)=(x(t),y(t),z(t))$ in the interval $t_0 \le t \le t_1$ is smooth if everywhere in this interval the derivative $\frac{d\mathbf{r}}{dt}$ exists.

Piecewise smooth

A piecewise smooth curve is made up of pieces of smooth curves joined end to end.



Simple open curve

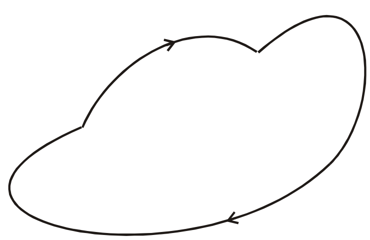
A simple open curve is one that does not cross itself. In this case the function

$\mathbf{r}(t)=(x(t),y(t),z(t))$

is one-one (a bijection). That is, each point of $\mathbf{r}(t)$ corresponds to just one value.

Simple closed curve

A simple closed curve is a curve whose end points coincide but all other points correspond to just one value of *t*.



Example (12)

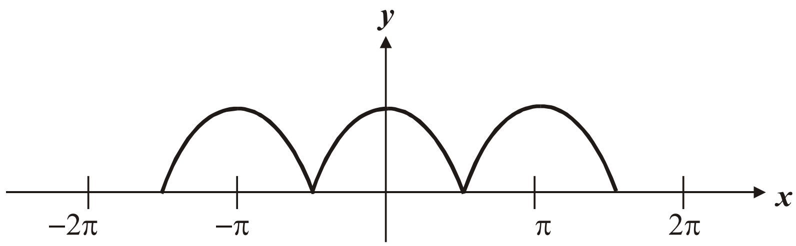
Sketch the curve with parametric equation

$\mathbf{r}(t)=(t,|\cos t|,0) \qquad -3\pi/2 \le t \le 3\pi/2$

and show that it is piecewise smooth.

<u>Solution</u>

The sketch is



As the sketch shows $\mathbf{r}(t)$ is made of three smooth curves

$\mathbf{r}_1(t) = (t, -\cos t, 0)$ $\quad -3\pi/2 \le t \le -\pi/2$

$\mathbf{r}_2(t) = (t, \cos t, 0)$ $\quad -\pi/2 \le t \le \pi/2$

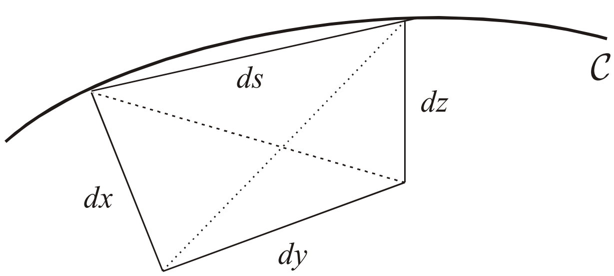
$\mathbf{r}_1(t) = (t, -\cos t, 0)$ $\quad \pi/2 \le t \le 3\pi/2$

Change of parameter

The curve $\mathbf{r} = \mathbf{r}(t)$ may be given a new parameter $t = t(s)$ where $\frac{ds}{dt} \neq 0$ at all points in the interval $s_0 \le t \le s_1$ and $t_0 = t(s_0), t_1 = t(s_1)$. Hence s increases as t increases, $t = t(s)$ is an increasing one-one function, and the direction (orientation) of $\mathbf{r} = \mathbf{r}(t)$ is preserved.

Arc length

Let $\mathbf{r} = \mathbf{r}(t)$ be the piecewise smooth curve *C*. Let *ds* be a small increase along the cord of this curve corresponding to small increases *dx*, *dy* and *dz*.



We have

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

and

$$\frac{ds}{dt} = \left|\frac{dr}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Then the arc length along C from t_0 (fixed) to t_1 (variable) is given by

$$\begin{aligned} s(t) &= \int_{t_0}^{t_1} \frac{ds}{dt} \cdot dt \\ &= \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \cdot dt \end{aligned}$$

Intrinsic equation of a curve

For the curve $\mathbf{r}(t) = (x(t), y(t), z(t))$

write the arc length from a (fixed) to t (variable) as

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

This gives s as a function of t, which we can solve to get t as a function of s. So we can reparametrize the curve $\mathbf{r}(t)$ by using s instead of t. This is the unit speed reparametrization of $\mathbf{r}(t)$. Hence, by using arc length as the parameter, we define the *intrinsic equation* of the curve.

$$\beta(s) = \mathbf{r}(t(s))$$

Then the unit tangent vector is obtained from this intrinsic equation as

$$\hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

Example (13)

Given $\mathbf{r} = (a\cos t, a\sin t, bt)$ $\qquad 0 \le t \le 2\pi$

(*a*) Find the intrinsic equation $\mathbf{r}(s)$ of the curve.

(*b*) Show that the derivative of $\mathbf{r}(s)$ is unitary.

<u>Solution</u>

(*a*) $\mathbf{r} = (a\cos t, a\sin t, bt)$

$$\frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = a\cos t \qquad \frac{dz}{dt} = b$$

$$s(t)=\int_{t_0}^{t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\ dt$$
$$=\int_{t_0}^{t_1}\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}\ dt$$
$$=\int_{t_0}^{t_1}\sqrt{a^2+b^2}\ dt$$
$$=\left(\sqrt{a^2+b^2}\right)t\Big|_{t_0}^{t_1}$$
$$=\left(\sqrt{a^2+b^2}\right)t \quad \text{where } t_0=0 \text{ and } t_1=t$$

Hence

$$t=\frac{1}{\sqrt{a^2+b^2}}s$$

Therefore, the intrinsic equation of the curve is

$$\mathbf{r}(s)=\left(a\cos t(s),a\sin t(s),bt(s)\right)$$
$$=\left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right),a\sin\frac{s}{\sqrt{a^2+b^2}},\frac{bs}{\sqrt{a^2+b^2}}\right)$$

(*b*) $$\mathbf{r}(s)=\left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right),a\sin\frac{s}{\sqrt{a^2+b^2}},\frac{bs}{\sqrt{a^2+b^2}}\right)$$
$$\mathbf{T}=\frac{d}{ds}\mathbf{r}(s)=\left(-\frac{a\sin s}{\sqrt{a^2+b^2}},\frac{a\cos s}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}}\right)$$
$$|\mathbf{T}|=\frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\sin^2 s+a^2\cos^2 s+b^2}=1$$

Example (14)

For the curve $\alpha(t)=\left(t,\frac{t^2}{\sqrt{2}},\frac{t^3}{3}\right)$

(a) Find the velocity, speed and acceleration for arbitrary t and at $t=1$.

(b) Find the arc-length function $s=s(t)$ based at $t=0$ and determine the arc length of α from $t=-1$ to $t=1$.

<u>Solution</u>

$$\alpha(t)=\left(t,\frac{t^2}{\sqrt{2}},\frac{t^3}{3}\right)$$

$\alpha'(t) = \left(1, \sqrt{2}\,t, t^2\right)$

$v(t) = \|\alpha'(t)\| = \sqrt{(1)^2 + \left(\sqrt{2}\,t\right)^2 + \left(t^2\right)^2} = \sqrt{1 + 2t^2 + t^4} = 1 + t^2$

$a(t) = \alpha''(t) = \left(0, \sqrt{2}, 2t\right)$

$v(1) = \alpha'(1) = \left(1, \sqrt{2}, 1\right)$

speed $= |v(1)| = |\alpha'(1)| = 2$

$\alpha''(1) = \left(0, \sqrt{2}, 2\right)$

$s(t) = \int_0^t |\alpha'(u)|\, du = \int_0^t 1 + u^2\, du = \left[u + \frac{u^3}{3}\right]_0^t = t + \frac{t^3}{3}$

Arc length from $t = -1$ to $t = 1$

$l = \left[u + \frac{u^3}{3}\right]_{-1}^{1} = \left(1 + \frac{1}{3}\right) - \left(-1 - \frac{1}{3}\right) = 2\frac{2}{3}$

Example (15)

Sketch the curve $\alpha(t) = (t\cos t, t\sin t, t)$. Find the velocity, speed and acceleration of $\alpha(t)$.

Solution

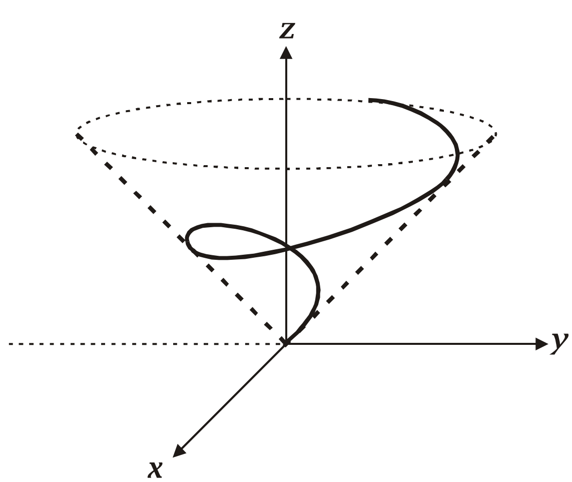
$\beta(t) = (t\cos t, t\sin t, 0)$ describes a circle in the *xy*-plane. A particle moving along this locus would be speeding up owing to the factor t and the circle would be steadily widening. In

$\alpha(t) = (t\cos t, t\sin t, t)$

the particle is also rising parallel to the *z*-axis at a uniform rate. Hence

$\alpha(t) = (t\cos t, t\sin t, t)$

describes a helix in Enclidean $\mathbb{E}^3$ space.



$\alpha(t) = (t\cos t, t\sin t, t)$

velocity $= \alpha'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$ at $\alpha(t)$

$$\begin{aligned}\text{speed} = |\alpha'(t)| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1} \\ &= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1} \\ &= \sqrt{2 + t^2}\end{aligned}$$

acceleration $= \alpha''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t, 0)$ at $\alpha(t)$

Example (16)

Show that the curve $\mathbf{r}(t) = (\cosh t, \sinh t, t)$ has arc-length function

$s(t) = \sqrt{2}\sinh t$ and find a unit-speed reparametrization of **r**.

$\mathbf{r}(t) = (\cosh t, \sinh t, t)$

$$\frac{d}{dt}\mathbf{r}(t) = (\sinh t, \cosh t, 1)$$

$$\text{speed} = \left|\frac{d}{dt}\mathbf{r}(t)\right| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{\sinh^2 t + \cosh^2 t + \cosh^2 t - \sinh^2 t} = \sqrt{2}\,\cosh t$$

$$s(t) = \int_0^t \sqrt{2}\cosh u\, du = \left[\sqrt{2}\sinh u\right]_0^t = \sqrt{2}\sinh t$$

Then

$$s = \sqrt{2}\sinh t \qquad \Rightarrow \qquad t = \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)$$

A unit speed reparametrization of **r** is given by

$$\begin{aligned}\beta(s) &= \mathbf{r}(t(s)) \\ &= \left(\cosh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \sinh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right) \\ &= \left(\sqrt{1 + \frac{s^2}{2}}, \frac{s}{\sqrt{2}}, \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right)\end{aligned}$$

This follows since $\cosh^2 x = 1 + \sinh^2 x$; hence

$$\cosh^2\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right) = 1 + \sinh^2\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right) = 1 + \frac{s^2}{2}$$

Example (17)

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be unit speed reparametrizations of the same curve $\mathbf{r}(t)$. Show that there is a number l such that $\mathbf{u}_2(s) = \mathbf{u}_1(s + l)$ for all s. Interpret the geometric significance of l.

Solution

Let

$\mathbf{u}_1(s) = \mathbf{r}(t(s))$

$\mathbf{u}_2(s^*) = \mathbf{r}(t(s^*))$

be two unit speed parametrizations of $\mathbf{r}(t)$. Then

$s = \int_{t_1}^{t} |\mathbf{r}'(u)| \, du$

$s^* = \int_{t_2}^{t} |\mathbf{r}'(u)| \, du$

Let $l = s^* - s = \int_{t_2}^{t} |\mathbf{r}'(u)| \, du - \int_{t_1}^{t} |\mathbf{r}'(u)| \, du = \int_{t_1}^{t_2} |\mathbf{r}'(u)| \, du =$ arclength from t_1 to t_2

Solving $l = s^* - s$ for s^*

$s^* = s + l = s +$ arclength from t_1 to t_2

Hence

$\mathbf{u}_2(s^*) = \mathbf{u}_1(s + l)$