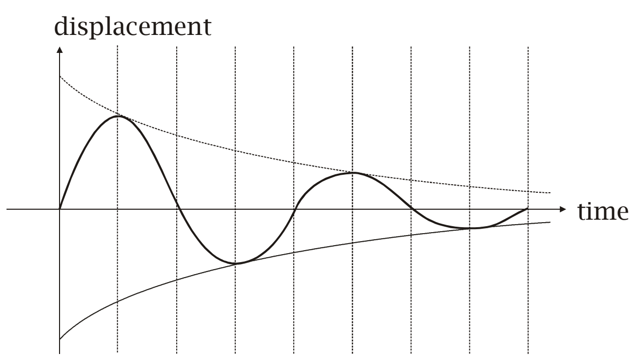
Damped Harmonic Motion

Prerequisites

You should already be familiar with the solution to homogeneous, constant coefficient second order differential equations, with simple harmonic motion and with sinusoidal functions.

Damped vibrations and linear resistive forces

Damped vibrations occur when the amplitude of an oscillating system progressively decreases.

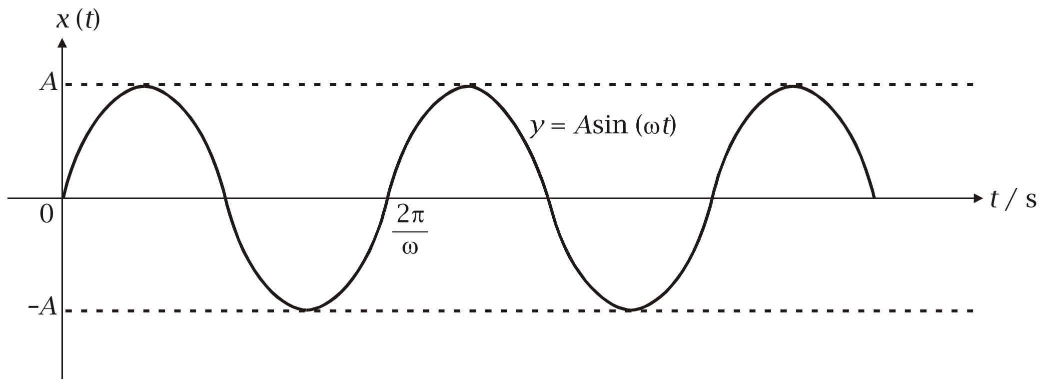


To study damped vibrations we must begin by reviewing undamped vibrations. Undamped oscillations occurs when an object oscillates under simple harmonic motion.

Simple harmonic motion

Simple harmonic motion is defined as motion taking place along a straight-line in which the acceleration $\left(\frac{d^2x}{dt^2}\right)$ of the object is proportional to the displacement (x) of the object from a fixed point and in the opposite direction to the displacement. The differential equation governing simple harmonic motion is $\frac{d^2x}{dt^2} = -\omega^2 x$ with solution $x = A \sin \omega t$.

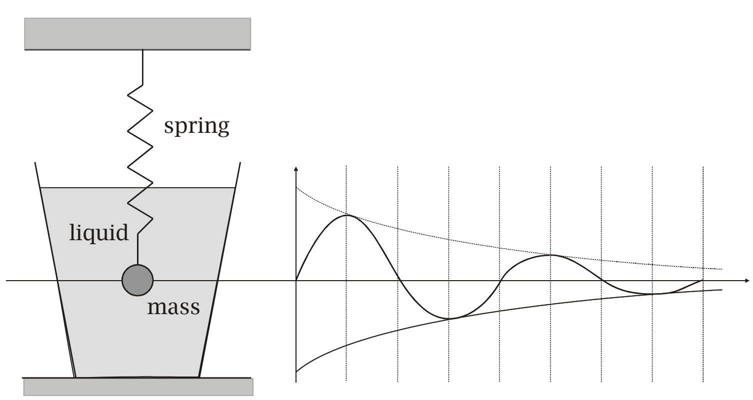
The graph of $x = A\sin\omega t$ is a simple sine wave with constant amplitude. Hence, the oscillations remain constant.



Damped oscillations occur when an equation takes the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

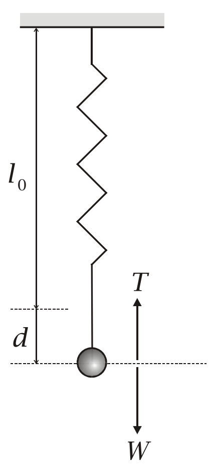
where *a*, *b* and *c* are positive constants. Comparing this to the case for undamped simple harmonic motion $\frac{d^2x}{dt^2} + \omega^2 x = 0$ we see that a term for the velocity of the particle, $\frac{dx}{dt}$, has appeared in the equation. Expressions like $\frac{dx}{dt}$ arise in situations where a particle is subject to *linear resistive forces*. It is these linear resistive forces that lead to the damping of the oscillation. A linear resistive force is a force acting on a particle whose magnitude is proportional to the speed at which the object is moving and whose direction opposes the velocity of the object. For example, suppose an object is suspended by a spring and is immersed in a bucket of a viscous liquid. It is displaced from its equilibrium position and starts to oscillate. However, the liquid progressively slows down the mass, so the amplitude of the oscillations progressively decreases. Consequently, the mass undergoes damped oscillation.



The motion of a mass oscillating in air will also be opposed by air resistance and so will undergo damped oscillations. (The point of the liquid in this example is to make the effect of damping more obvious.) We shall now show how a system of this kind can lead to a differential equation of the form

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$$

Let the mass of the particle be *m*, let the stiffness of the spring be *k*, let the natural length of the spring be l_0 and let the magnitude of the resistive force of the treacle be $R=r\frac{dx}{dt}$ where *r* is a constant and $v(t)=\frac{dx}{dt}$ is the velocity of the mass. The resistive force is understood to oppose the motion of the particle. We will suppose the spring is displaced by $x=x(t)$ from its equilibrium position. However, as usual, we start to study such systems, by considering what happens at the equilibrium position first. When the mass hangs under its own weight in the equilibrium position the forces may be drawn as follows.



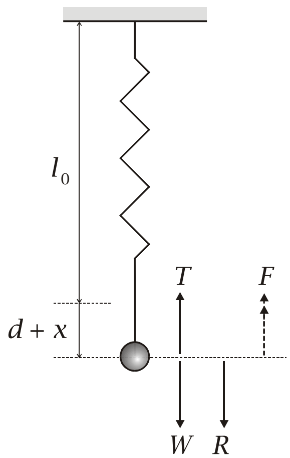
The mass is subject to two forces, the tension in the spring pulling it upwards and the weight pulling it downwards. The tension is given by *kd* where *d* is the extension of the spring up to the equilibrium point. The mass is given by $W=mg$. Since the system is in equilibrium

$$T=W$$

$$kd=mg$$

Now suppose that the mass is displaced from the equilibrium position by a further $x=x(t)$. It is then subject to three forces. The tension (T) due to the extension of the spring. Its weight

$(W = mg)$ and the linear resistive force $\left(R = r\frac{dx}{dt}\right)$ arising from the opposition to motion of the fluid or air.



In the diagram above we imagine that the mass is about to move upwards. As the linear resistive force $\left(R = r\frac{dx}{dt}\right)$ opposes the motion, we show it pointing downwards. The total extension is $d + x$ and the tension is

$$T = -k(d + x)$$

The negative sign indicates that the tension is in the opposite direction to the displacement. The weight is, as usual

$$W = mg$$

The *linear resistive force* is

$$R = -r\frac{dx}{dt}$$

It acts as to oppose the motion, which is shown by the negative sign. The resultant force acting on the particle is

$$F = T + R + W$$

and by Newton's second law this is

$$F = ma = m\frac{d^2x}{dt^2}$$

Hence

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -k(d + x) - r\frac{dx}{dt} + mg \\ &= -kd - kx - r\frac{dx}{dt} + mg \end{aligned}$$

Since $kd = mg$ we have

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - kx$$

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0$$

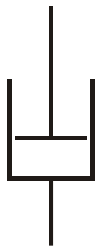
which is a second order homogeneous constant coefficient differential equation.

Second order homogeneous constant coefficient differential equations

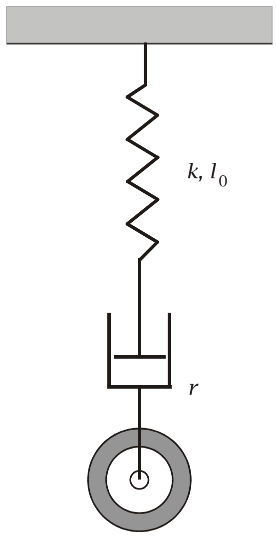
Given $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ where a, b and c are constants, the auxiliary equation is $am^2 + bm + c = 0$ with roots m_1 and m_2 and discriminant $\Delta = b^2 - 4ac$. If $\Delta > 0$ the roots m_1 and m_2 are real and distinct, the solution to the original differential equation takes the form $x(t) = Ae^{m_1 t} + Be^{m_2 t}$ where A and B are constants. If $\Delta = 0$ then $m_1 = m_2$ and the root is real and repeated, the solution to the original differential equation takes the form $x(t) = (A + Bt)e^{m_1 t}$ where A and B are constants. If $\Delta < 0$ the roots m_1 and m_2 are conjugate complex numbers where $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the solution to the original differential equation takes the form $x(t) = e^{\alpha t}(A\sin\beta t + B\cos\beta t)$ where A and B are constants.

Dashpots

The purpose of a car suspension system is to bring about the damping of oscillations in a tyre as it runs over a bump. We will be initially considering the case where the tyre runs over a single bump, which effectively gives it a sudden sharp displacement from its equilibrium position. In order to bring about the damping the suspension system is fitted with a mechanical device called a *dashpot*. The dashpot is represented thus



Its function is to provide a linear resistive force $R = -r\frac{dx}{dt}$ where r is the dashpot constant. The whole car suspension system can be represented by the following.



In this application the oscillating mass is provided by the chassis, and we are imagining that the tyre, dashpot and spring have no mass. This creates complications in the mathematics that we wish to avoid at this stage, so we will continue to deal with the solution to questions where the oscillating mass is given by the object here labelled the tyre, and the object labelled by the chassis is fixed. The symmetry of the situation means that what applies to a tyre of mass m where the chassis is fixed and has no mass would also apply to a chassis of mass m where the tyre is fixed and has no mass.

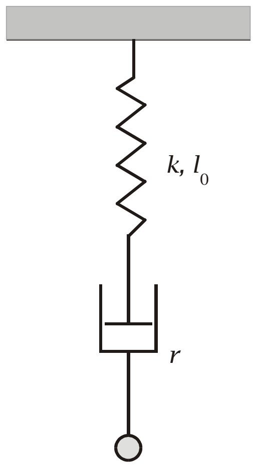
In other mechanical systems where there is a linear resistive force, this force may also be represented by dashpot symbol.

Example (1)

A mass, m of 2 kg, is suspended by a spring of natural length $l_0 = 2\text{m}$ and stiffness $k = 5$ and by a dashpot with dashpot constant $r = 2$. It is subject to a sudden sharp displacement of 0.5 m. Find its equation of motion and its phase lag. Sketch a graph showing the subsequent motion. Find its maximum amplitude subsequent to the displacement, its angular frequency, frequency and period.

Solution

The system can be represented as follows.



Let $x = x(t)$ represent its displacement at time t after it receives the shock, so $x(0) = 0.5$.

The general equation of motion for this system is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0$$

Here $m = 2,\ r = 2,\ k = 5$. Hence

$$2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

We now proceed to solve this equation. The auxiliary equation is

$$2m^2 + 2m + 5 = 0$$

$$\begin{aligned} m &= \frac{-5 \pm \sqrt{25-34}}{2} \\ &= \frac{-1 \pm \sqrt{-9}}{2} \\ &= \frac{-1 \pm 3i}{2} \\ &= -0.5 \pm 1.5i \end{aligned}$$

The solution is complex which indicates that there are oscillations. The general solution is

$$x(t) = e^{-0.5t}\left(A\sin(1.5t) + B\cos(1.5t)\right)$$

We need to determine the constants A and B. We have initial conditions $x(0) = 0.5$ and also $v(0) = \frac{dx}{dt}(0) = 0$ since it is not moving at $t = 0$. Then $x(0) = 0.5$ implies $B = 0.5$.

Differentiating $x(t)$ gives

$$v(t) = \frac{dx}{dt} = -0.5e^{-0.5t}\left(A\sin(1.5t) + B\cos(1.5t)\right) + e^{-0.5t}\left(1.5A\cos(1.5t) - 1.5B\sin(1.5t)\right)$$

Substituting $v(0) = 0,\ t = 0$

$$0 = -0.5B + 1.5A$$

Since $B = 0.5$

$$0 = -0.25 + 1.5A$$

$$A = \frac{0.25}{1.5} = \frac{1}{6}$$

Therefore, the equation of motion is

$$x(t) = e^{-0.5t}\left(\frac{1}{6}\sin(1.5t) + \frac{1}{2}\cos(1.5t)\right)$$

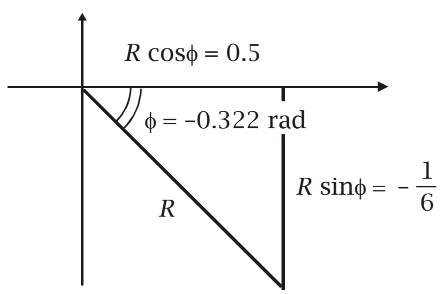
The presence of both a sine and cosine element in the part determining the oscillations indicates that there is a phase lag. Let

$$\frac{1}{6}\sin(1.5t) + \frac{1}{2}\cos(1.5t) = R\cos(1.5t + \phi)$$

where ϕ is the phase lag. Then

$$\frac{1}{6}\sin(1.5t) + \frac{1}{2}\cos(1.5t) = R\left(\cos(1.5t)\cos\phi - \sin(1.5t)\sin\phi\right)$$

$$R\cos\phi = \frac{1}{2} \qquad R\sin\phi = -\frac{1}{6}$$



$$\phi = \tan^{-1}\left(\frac{-\left(\frac{1}{6}\right)}{\left(\frac{1}{2}\right)}\right) = -\frac{1}{3}... = -0.322 \text{ rad} \quad (3 \text{ s.f.})$$

$$R = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2} = 0.52704... = 0.527 \quad (3 \text{ s.f.})$$

So the equation of motion can also be written

$$x(t) = e^{-0.5t}\left(0.527\cos(1.5t - 0.322)\right)$$

The phase lag is given by

$$1.5t - 0.32175... = 0$$
$$t = 0.215 \text{ s} \quad (3 \text{ s.f.})$$

So the phase lag of 0.322 rad corresponds to a time of 0.215 s.

The angular velocity of the damped oscillation is $\omega = 1.5 \text{ rad s}^{-1}$. The frequency is

$$f = \frac{\omega}{2\pi} = \frac{1.5}{2\pi} = 0.239 \text{ Hz} \;\; (3 \text{ s.f.})$$

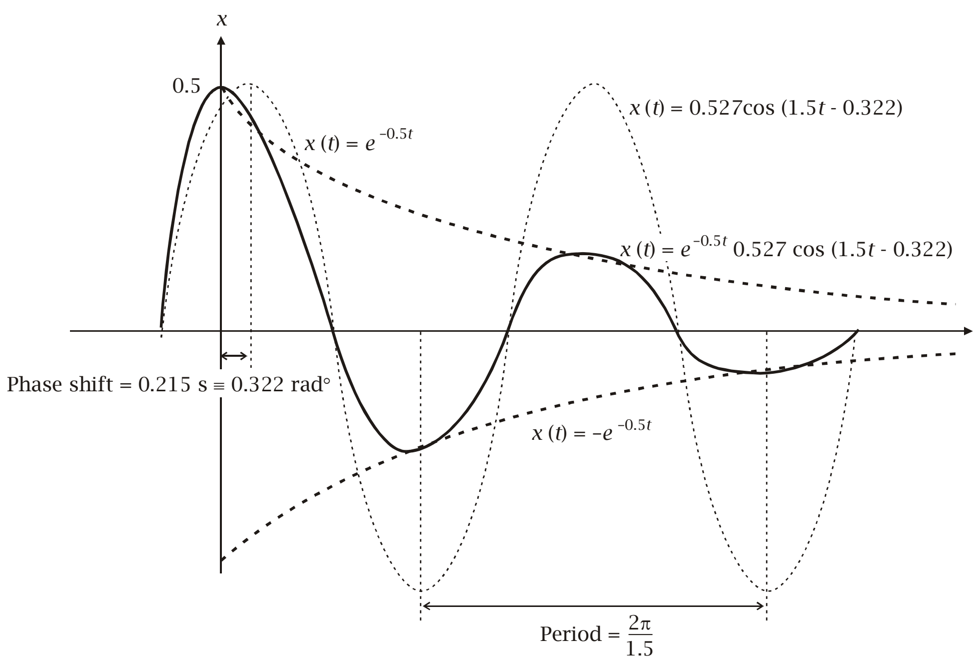
The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1.5} = 4.19 \text{ s} \;\; (3 \text{ s.f.})$$

In every half cycle the velocity is zero and the amplitude is at a maximum. At $t = 0$ the amplitude is 0.5 m due to the initial displacement. At $\frac{T}{2} = \frac{\pi}{1.5}$ s the velocity will again be zero, at which point the displacement will be

$$x(t) = e^{-0.5\left(\frac{\pi}{1.5}\right)}\left(\frac{1}{6}\sin(0) + \frac{1}{2}\cos(-\pi)\right) = -0.175 \text{ m} \;\; (3 \text{ s.f.})$$

A sketch of the graph of $x(t) = e^{-0.5t}\left(0.527\cos(1.5t - 0.322)\right)$ is as follows.



Regarding this sketch note that because of the damping factor the maxima of $x(t) = e^{-0.5t}\left(0.527\cos(1.5t - 0.322)\right)$ occur $\phi = 0.322 \text{ rad} \equiv 0.215 \text{ s}$ before the maxima of the corresponding undamped function $x(t) = 0.527\cos(1.5t - 0.322)$. The graph of the damping function $x(t) = e^{-0.5t}$ is not in fact tangent to $x(t) = e^{-0.5t}\left(0.527\cos(1.5t - 0.322)\right)$.

Critical and non-critical damping

We have seen that the general equation of motion for a system subject to linear resistive force and consequently subject to damping is

$$m\frac{d^2x}{dt^2}+r\frac{dx}{dt}+kx=0 \qquad m=\text{mass},\ r=\text{constant of linear resistance},\ k=\text{spring constant}$$

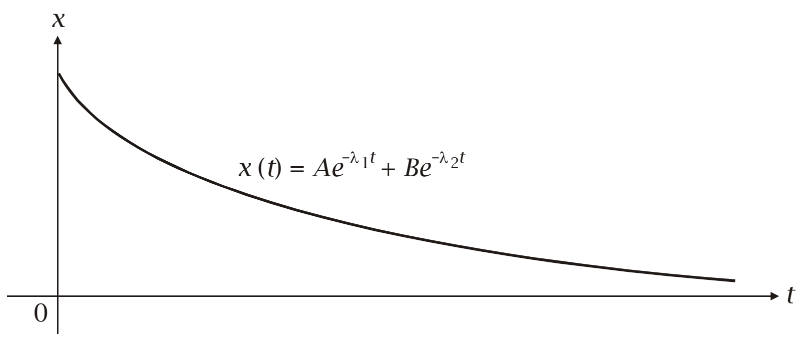
The theory of homogenous second order constant coefficient differential equations tells us that this has solutions depending on the nature of the roots to the auxiliary equation

$$mx^2+rx+k=0$$

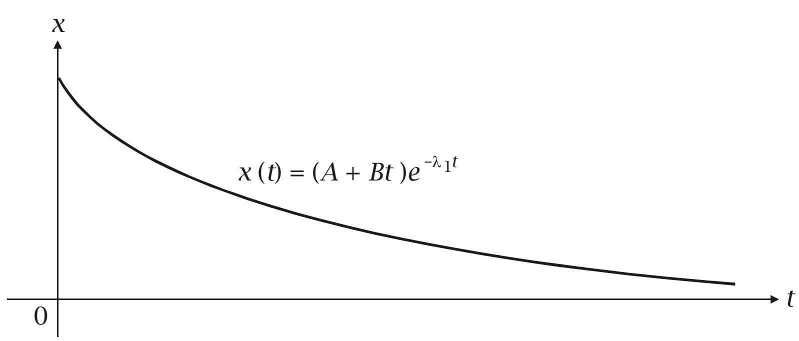
Suppose λ_1,λ_2 are the two roots of this equation. Then, there are three cases:

(1) λ_1,λ_2 are real and distinct $(\lambda_1 \neq \lambda_2)$ then $x(t)=Ae^{\lambda_1 t}+Be^{\lambda_2 t}$

In the case where there is a linear resistive force the roots λ_1 and λ_2 will both be negative. This indicates a situation where there is *overdamping*. The system does not oscillate but experiences an immediate exponential decrement returning it to the equilibrium position.



(2) If $\lambda_1=\lambda_2$ is a real repeated root, the solution takes the form $x(t)=(A+Bt)e^{\lambda_1}$. The system does not travel beyond the equilibrium position, and is brought back to the equilibrium position. The motion is not oscillatory.



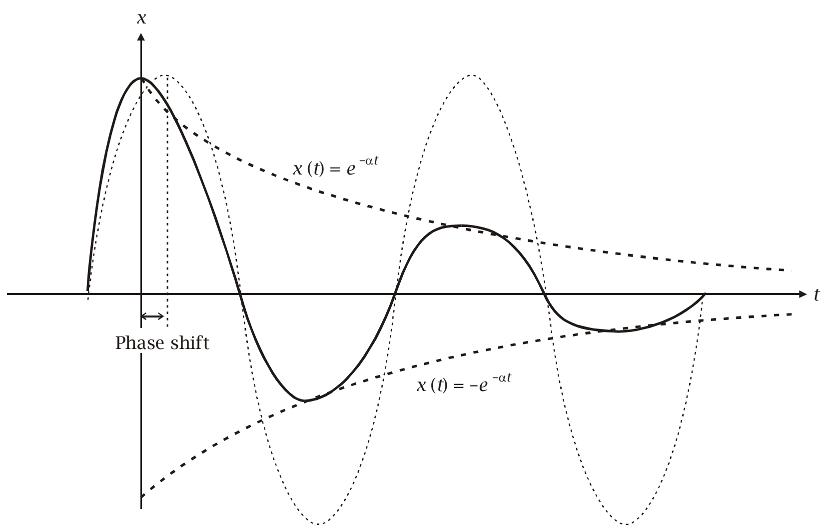
(3) If λ_1,λ_2 are complex conjugate numbers so that

$$\lambda_1 = \alpha + i\beta$$
$$\lambda_2 = \alpha - i\beta$$

then the solution is

$$x(t) = e^{\alpha t}\left(A\sin\beta t + B\cos\beta t\right)$$

Again the requirement that the linear resistive force acts in the opposite direction to the velocity entails that α is a negative quantity. Hence, the system oscillates but the amplitude of the oscillations is damped.



As the graph indicates, generally we expect a phase lag.

The criterion for critical damping

For oscillations to take place there must be complex roots. Examining the equation

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \qquad m = \text{mass},\ r = \text{constant of linear resistance},\ k = \text{spring constant}$$

with the auxiliary equation

$$mx^2 + rx + k = 0$$

We require that the discriminant

$$\Delta = \sqrt{r^2 - 4mk} < 0$$
$$4mk > r^2$$
$$\frac{r^2}{4mk} < 1$$

Let us define

$$\alpha = \sqrt{\frac{r^2}{4mk}} = \frac{r}{2\sqrt{mk}}$$

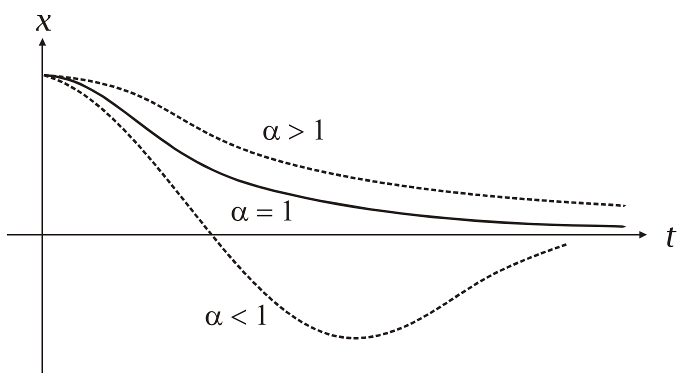
to be the damping factor. The requirement for oscillations to take place is, therefore,

$\alpha < 1$

When $\alpha = 1$ we have the case of the single repeated root

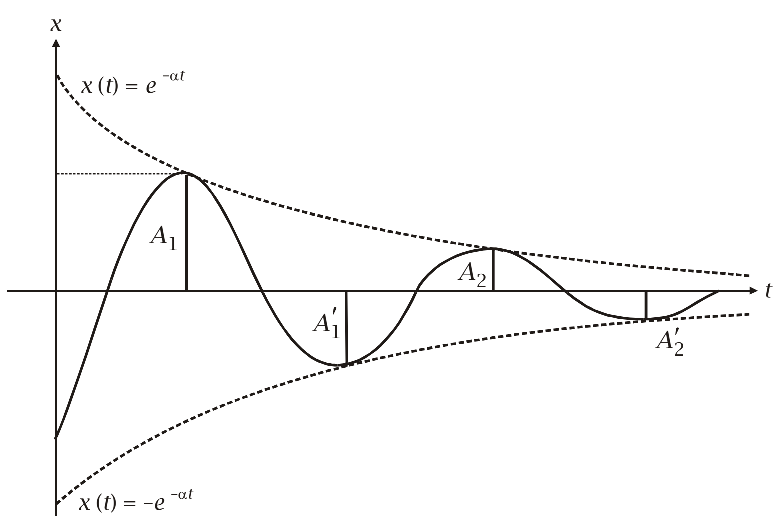
$$\lambda = -\frac{r}{2m}$$

with solution $x(t) = (A + Bt)e^{\lambda t}$. This situation, where $\alpha = 1$, is called *critical damping*. It returns the system to the equilibrium position at the fastest possible rate without overshooting the equilibrium position. When such things as gun recoil mechanisms are designed, it is this that is selected.



Amplitude of successive oscillations

Consider the case where the damping factor $\alpha < 1$ so there are damped oscillations.



The diagram marks the successive amplitudes of the oscillations. The general solution is

$x(t) = e^{at}\left(A\cos(\beta t + \phi)\right)$

where $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are the two complex conjugate solutions to the equation

$$mx^2 + rx + k = 0$$

Hence

$$\lambda_{1,2} = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}$$

and

$$\alpha = \frac{-r}{2m} \text{ and } \beta = \frac{\sqrt{r^2 - 4mk}}{2m}$$

The angular frequency is given by β so the period is $T = \frac{2\pi}{\beta}$. Successive peaks occur when $\cos(bt + \phi) = 1$ i.e. when $bt + \phi = 0$. This occurs after every complete oscillation – that is, after intervals of the period *T*. Let A_1 represent the first peak, then at

$$A_1 = Ae^{\alpha t_1}$$

where t_1 is the first value such that

$$bt + \phi = 0$$

Then the second peak occurs at

$$t = t_1 + T$$

and the *n*th at $t = t_1 + nT$. Hence

$$A_n = Ae^{\alpha(t_1 + (n-1)T)} \quad \text{and} \quad A_{n+1} = Ae^{\alpha(t_1 + nT)}$$

Hence the ratio

$$\begin{aligned}\frac{A_n}{A_{n+1}} &= \frac{Ae^{\alpha(t_1+(n-1)T)}}{Ae^{\alpha(t_1+nT)}} \\ &= e^{\alpha(t_1+(n-1)T) - \alpha(t_1+nT)} \\ &= e^{-T}\end{aligned}$$

So successive peaks maintain a constant ratio. That is, the amplitude of successive oscillations decreases with time in geometric progression with common ratio e^{-T}.