De Moivre's Theorem

Prerequisites

You should be familiar with the various ways of representing a complex number in Cartesian form, in polar (trigonometric) form and in exponential form.

$$
\begin{aligned}
z &= [|z|, \arg z] \\
&= [r, \theta] \\
&= r(\cos\theta + i\sin\theta) \\
&= (x, y) \\
&= x + iy \\
&= re^{i\theta} \\
&= |z|e^{i\arg z}
\end{aligned}
$$

The first three here are three forms of the polar representation of z; the next two are Cartesian forms, the last two are exponent forms. To understand this chapter you also require knowledge of mathematical induction.

De Moivre's theorem

De Moivre's theorem is a result that enables us to find powers and roots of complex numbers. It tells us how to evaluate powers of a complex number – that is, how to find z^n. It can be expressed in Cartesian and polar (trigonometric) form.

De Moivre's theorem – Cartesian form

$$z^n = r(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

De Moivre's theorem – Polar form

$$z^n = [r, \theta]^n = [r^n, n\theta]$$

Example (1)

Express $\left(2\cos\frac{\pi}{8} + 2i\sin\frac{\pi}{8}\right)^2$ in the form $x + iy$.

Solution

$$\left(2\cos\frac{\pi}{8}+2i\sin\frac{\pi}{8}\right)^2=\left[2,\frac{\pi}{8}\right]^2 \qquad \left[\text{Putting } z \text{ in polar form}\right]$$
$$=\left[2^2,2\times\frac{\pi}{8}\right] \qquad \left[\text{Applying De Moivre's theorem}\right]$$
$$=\left[4,\frac{\pi}{4}\right]$$
$$=\left(4\cos\frac{\pi}{4}+i4\sin\frac{\pi}{4}\right) \qquad \left[\text{Returning to Cartesian form}\right]$$
$$=2\sqrt{2}+2\sqrt{2}\,i$$

Proof of De Moivre’s Theorem

The proof of De Moivre’s theorem follows by mathematical induction and exploits the property of multiplication of complex numbers. In polar form this is

$[r_1,\theta_1][r_2,\theta_2]=[r_1r_2,\theta_1+\theta_2]$

The proof in polar form is particularly straightforward and elegant.

Proof of De Moivre’s Theorem

To prove

$z^n=[r,\theta]^n=[r^n,n\theta]$

Proof by mathematical induction.

For the particular step, when $n=1$ $\quad [r,\theta]^1=[r^1,1\times\theta]$

For the induction step the induction hypothesis is

For $n=k$ $[r,\theta]^k=[r^k,k\theta]$

$[r,\theta]^k=[r^k,k\theta]$

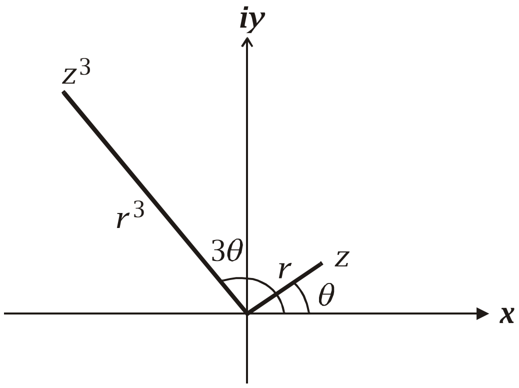
To prove for $n=k+1$ $[r,\theta]^{k+1}=[r^{k+1},(k+1)\theta]$. Now

$$[r,\theta]^{k+1}=[r,\theta][r,\theta]^k$$
$$=[r,\theta][r^k,k\theta] \qquad \left[\text{By the induction hypothesis}\right]$$
$$=[r\times r^{k+1},\theta+k\theta] \qquad \left[\text{Multiplication of complex numbers}\right]$$
$$=[r^{k+1},(k+1)\theta]$$

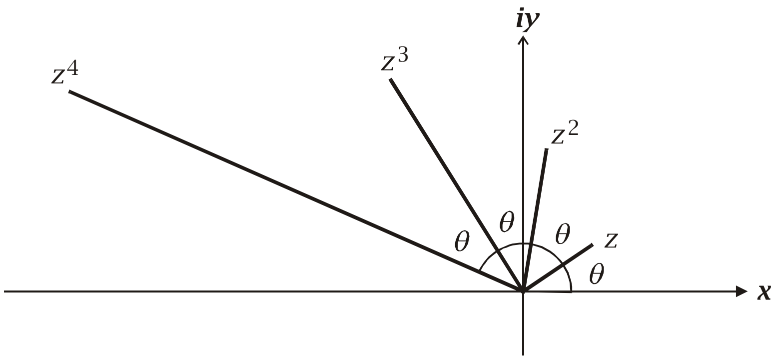
Hence the induction step holds and the result is true for all *n*. Converting into Cartesian form gives: $z^n=r(\cos\theta+i\sin\theta)^n=r^n\left(\cos n\theta+i\sin n\theta\right)$

Interpretation of De Moivre's Theorem and the *n* roots of unity

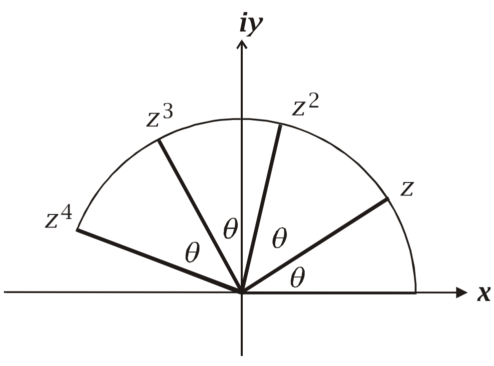
Suppose that $z = [r, \theta]$. For a definite illustration let us consider $z^3 = [r^3, 3\theta]$. Then graphically we plot z^3 by noting (1) that the argument of z^3 is 3 times the argument of *z*; (2) that the modulus of z^3 is the cube of the modulus of *z*.



If $r > 1$ then the values of $z^2, z^3, z^4, \ldots$ "spiral outwards".



If $r < 1$ then the values $z^2, z^3, z^4, \ldots$ "spiral inwards". Whilst if $r = 1$ then the values of $z^2, z^3, z^4, \ldots$ all lie on the unit circle.



The previous illustration suggests that we can apply De Moivre's theorem in reverse to find solutions to the equation $z^n = 1$. This is indeed the case. We observe that the equation $x^2 = 1$ has two solutions, $x = i$ and $x = -i$. Likewise, we expect the equation

$$z^n = 1$$

to have n solutions, and this is the case. In polar form the equation $z^n = 1$ takes the form

$$[r, \theta]^n = [1, 0]$$

Applying De Moivre's theorem we get

$$[r^n, n\theta] = [1, 0]$$

Hence $r^n = 1$ and $n = 1$ and $n\theta = 0$. One solution to the equation $n\theta = 0$ is $\theta = 0$. However, we should recall that the angle 0 is given modulo 2π and that

$$0 \equiv 2\pi = 4\pi \equiv \ldots \equiv 2n\pi \equiv \ldots \quad (\mathrm{mod}\, 2\pi)$$

Hence the n roots of unity – that is the n roots to the equation $z^n = 1$ are given by the n distinct solutions to the equation $n\theta \equiv 0 (\mathrm{mod}\, 2\pi)$

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots.$$

The solutions in polar form are the n distinct complex numbers

$$[1, 0], \left[1, \frac{2\pi}{n}\right], \left[1, \frac{4\pi}{n}\right], \left[1, \frac{6\pi}{n}\right], \ldots$$

Example (2)

Solve $z^3 = 1$

Solution

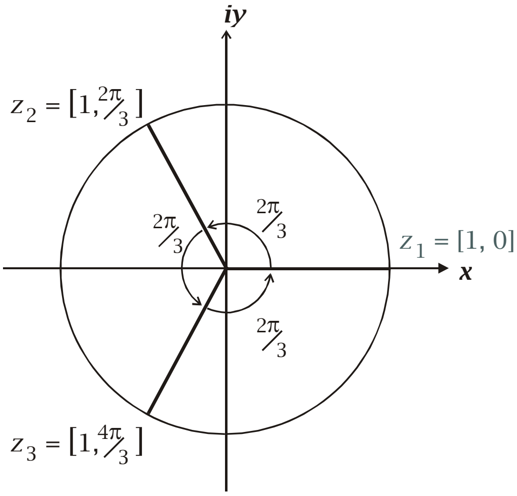
By substitution of $n = 3$ into the formula

$$[1, 0], \left[1, \frac{2\pi}{n}\right], \left[1, \frac{4\pi}{n}\right], \left[1, \frac{6\pi}{n}\right], \ldots$$

the solutions are

$$[1, 0], \left[1, \frac{2\pi}{3}\right], \left[1, \frac{4\pi}{3}\right]$$

Graphically, these solutions are represented as follows.



In Cartesian form

$z_1 = \cos 0 + i\sin 0 = (1,0)$ $\qquad$ $z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$z_3 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

We can also use De Moivre's theorem to find solutions to equations such as $z^4 = -1$.

Example (3)

Solve $z^4 = -1$.

Solution

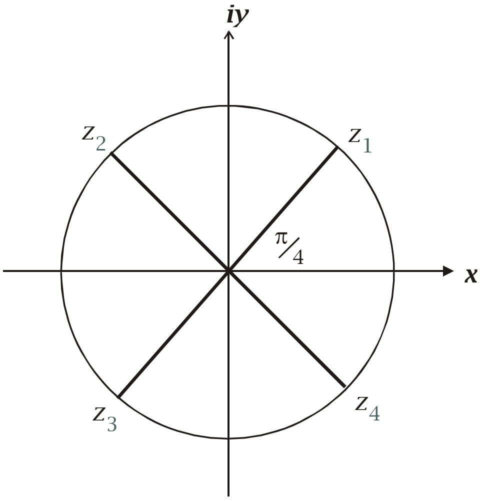
$[1,\theta]^4 = [1,\pi]$

$[1,4\theta] = [1,\pi]$

$\therefore 4\theta = \pi \quad (\mathrm{mod}\, 2\pi)$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$z_1 = \left[1, \frac{\pi}{4}\right]$ $\qquad$ $z_2 = \left[1, \frac{3\pi}{4}\right]$ $\qquad$ $z_3 = \left[1, \frac{5\pi}{4}\right]$ $\qquad$ $z_4 = \left[1, \frac{7\pi}{4}\right]$



In Cartesian form

$z_1 = \cos \pi/4 + i \sin \pi/4 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

$z_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ $\qquad$ $z_3 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$ $\qquad$ $z_4 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$

Applications of De Moivre's theorem to trigonometric identities

By expanding $(\cos\theta + i\sin\theta)^n$ using the Binomial theorem (or Pascal's triangle) and equating with $\cos n\theta + i \sin n\theta$ we can obtain further trigonometric identities. Recall that De Moivre's theorem is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$

Since the real and imaginary parts of both sides of this equation are independent of each other, we can equate real and imaginary parts to obtain trigonometric identities. The whole process is best grasped through illustration.

Example (4)

Prove $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

Solution

By De Moivre's theorem

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Pascal's triangle up to $n = 5$ gives

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Hence

$$
\begin{aligned}
\cos 5\theta + i\sin 5\theta = {} & \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta \\
& +10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta
\end{aligned}
$$

Since $i^2 = -1$ we have

$$
\begin{aligned}
\cos 5\theta + i\sin 5\theta = {} & \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
& -10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta
\end{aligned}
$$

On equating real parts and using the identity $\cos^2\theta + \sin^2\theta \equiv 1$ we get

$$
\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
&= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\
&= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta) \\
&= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}
$$

By equating imaginary parts we can also show

$$
\begin{aligned}
\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\
&= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
&= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
&= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta
\end{aligned}
$$