Derivatives of Exponential and Logarithmic Functions

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Derivative of the exponential function

The exponential function $y = e^x$ has the unique property that its derivative is identical to

$\frac{d}{dx}e^x = e^x$ If $f(x) = e^x$ then $f'(x) = e^x$

Other related derivatives are

$f(x) = e^{ax}$ $f'(x) = ae^{ax}$

$f(x) = e^{-x}$ $f'(x) = -e^{-x}$

$f(x) = 2^x$ $f'(x) = \ln 2 \times 2^x$

Derivative of the logarithmic function

The derivative of logarithm to the base e (ln x) is

$\frac{d}{dx}\ln x = \frac{1}{x}$ If $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$

Also, $f(x) = \ln(ax)$ then $f'(x) = \frac{1}{x}$

Example

Differentiate

(a) $\frac{e^{x^2} - e^{-x^2}}{3}$

(b) e^{ax^2+bx+c}

(c) $\log(\sqrt{\sin x})$

Solution

(a) $\frac{d}{dx}\left(\frac{e^{x^2} + e^{-x^2}}{3}\right) = \frac{2xe^{x^2} - 2xe^{-x^2}}{3}$

(b) $\left(e^{ax^2+bx+c}\right)' = e^{ax^2+bx+c} \times (2ax+b) = (2ax+b)e^{ax^2+bx+c}$

(*c*) $\left(\log\left(\sqrt{\sin x}\right)\right)' = \frac{1}{\sqrt{\sin x}} \times \frac{1}{2\sqrt{\sin x}} \times \cos x = \frac{\cos x}{2\sin x} = \frac{1}{2}\cot x$

Sketching the curves of exponential functions

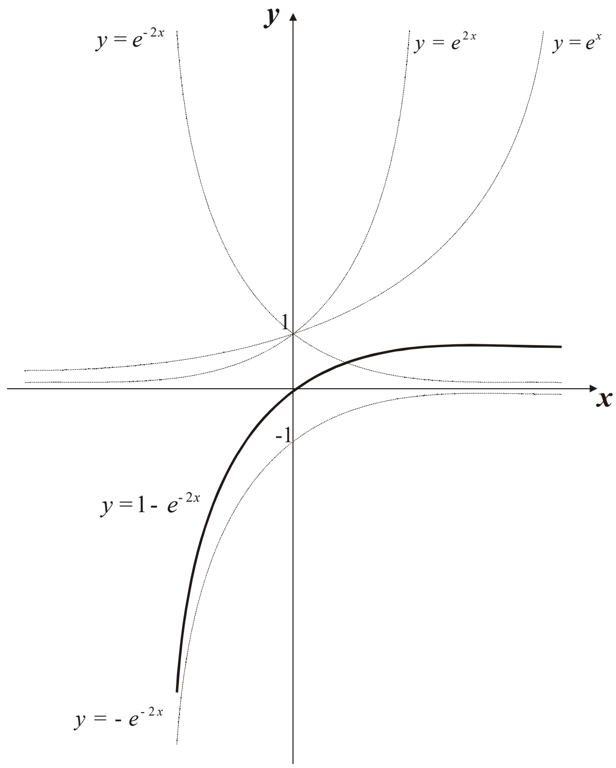
Sketching the curves of exponential functions follows the usual rules for the transformation of graphs.

Example

Sketch the curve $y = 1 - e^{-2x}$

Solution

We start with the function $y = e^x$ and scale it vertically by a factor of 2 to obtain the function $y = e^{2x}$. We reflect this in the vertical axis to obtain $y = e^{-2x}$. We reflect this in the horizontal axis to obtain $y = -e^{-2x}$. Finally, we translate this vertically by +1 to obtain $y = 1 - e^{-2x}$.



The Number e

The number e may be defined as the real number such that $\frac{d}{dx}e^x = e^x$. However, there is another, arguably more fundamental, approach to the definition of e, that derives from its role in the calculation of compound interest.

Compound interest concerns the question of repeatedly calculating interest by charging a rate added back to the principle invested.

Example
Find £250 invested for 3 years at 8% per annum.
Solution
Let us do this first by the "slow method".
8% of 250 = $250 \times 0.08 = 20$.
Value after 1st year = 250 + 20 = £270.
This is equivalent to $250 \times 1.08 = 270$.
Repeating this process:
After the 2nd year: $270 \times 1.08 = 291.60$
After the 3rd year: $291.6 \times 1.08 = 314.928$
Hence after 3rd years the sum is £314.93 to the nearest penny.
This can all be done in one step:
$291.6 \times (1.08)^3 = 314.928$

Hence, we calculate the process in three **discrete** stages. However, we wish to extend this kind of operation to a **continuous** process. That is, the idea of continuously charging compound interest.

Fundamentally, this is the same problem as finding, for a given rate of interest, the value of a principle taken over a single time period – for example, 100% compound interest charged continuously for 1 year. We argue as follows:

100% compound interest over one year.
This is equivalent to doubling the value of the principle, so at the end of the period, the value is:

$P = P_0(1+1)$, where P_0 is the original sum invested. In the following, let $P_0 = 1$.

50% compound interest charged twice over a given year:

$$P = P_0\left(1+\frac{1}{2}\right)^2 = 1+1+\frac{1}{4} = \frac{9}{4}.$$

33.33...% compound interest charged three times over a given year:

$$P = P_0\left(1+\frac{1}{3}\right)^3 = 1+1+\frac{1}{3}+\frac{1}{27} = \frac{64}{27}.$$

If we continue this process, we reach the idea of $\frac{100}{n}\%$ interest charged n times over a single time period (here one year). $P = P_0\left(1+\frac{1}{n}\right)^n$. The Binomial theorem for rational index is:

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+....$$

Hence:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+n\frac{1}{n}+\frac{n(n-1)\left(\frac{1}{n}\right)^2}{2!}+\frac{n(n-1)(n-2)\left(\frac{1}{n}\right)^3}{3!}+....\\ &= 1+1+\frac{1}{2!}-\frac{1}{2!n}+\frac{1}{3!}-\frac{1}{3!n}+\frac{2}{3!n^2}+...\\ &= 1+1+\frac{1}{2!}+\frac{1}{3!}+...-\frac{1}{2!n}-\frac{1}{3!n}-...+\frac{2}{3!n^2}+\end{aligned}$$

The idea of charging interest continuously is equivalent to letter $n \to \infty$ in this expression. Then all the terms with n in the denominator tend to zero (we say "vanish"), and we obtain for 100% interest charged continuously over one time period:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+... \approx 2.71828.....$$

This is the definition of the number e.

We can extend these ideas to the notion of charge a rate r, not necessarily equivalent to 100% interest, on n occasions in each time period for t time periods. Let P_0 be the principle, r the rate, and t the number of time periods to calculate. Then after t periods the value is:

$$P = P_0\left(1+\frac{r}{n}\right)^{nt}.$$

Making this into a continuous variable in n gives:

$$P = P_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}.$$

To show the connection with the number e, let us make the substitution $x = \frac{1}{\left(r/n\right)} = \frac{n}{r}$. This also gives $n = xr$. Then,

$$P = P_0 \lim_{n\to\infty 0}\left(1+\frac{1}{x}\right)^{xrt} = P_0\left(\lim_{n\to\infty 0}\left(1+\frac{1}{x}\right)^{x}\right)^{rt} = P_0 e^{rt}.$$

So we see that e is fundamental to all calculations involving continuous compound interest. Since in applications in physics the idea of a continuously varying physical quantity, for example, the decrease in temperature of a standing bowl of initially hot water, crops up in every context, we see that the number e is fundamental not only to business, but also to our modelling of the real world.

Proofs of the derivatives of ln*x* and *e*^{x}

So we **define** e to be: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots \approx 2.71828....$. Given the exponential function: $f(x) = e^x$ and its inverse, the natural logarithm, $f^{-1}(x) = \ln x$, we now seek to **prove** the two fundamental properties of their calculus. Namely:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

To do this, we start with the formula for the natural logarithm.

Theorem

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

<u>Proof</u>

From the definition of the derivative:

$$\frac{d}{dx}\ln x = \lim_{\delta x \to 0}\left\{\frac{\ln(x+\delta x) - \ln x}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0}\left\{\frac{\ln\left(\frac{x+\delta x}{x}\right)}{\delta x}\right\} = \lim_{\delta x \to 0}\left\{\frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)\right\} = \lim_{\delta x \to 0}\left\{\ln\left(1+\frac{\delta x}{x}\right)^{\frac{1}{\delta x}}\right\}$$

Substitute $u = \frac{\delta x}{x} \Rightarrow xu = \delta x \Rightarrow \frac{1}{\delta x} = \frac{1}{xu}$; also $\delta x \to 0 \Rightarrow u \to 0$. Then:

$$\frac{d}{dx}\ln x = \lim_{\delta x \to 0}\left\{\ln\left(1+\frac{\delta x}{x}\right)^{\frac{1}{\delta x}}\right\}$$

$$= \lim_{\delta x \to 0}\left\{\ln(1+u)^{\frac{1}{xu}}\right\}$$

$$= \lim_{\delta x \to 0}\left\{\left(\ln(1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}}\right\} = \lim_{\delta x \to 0}\left\{\frac{1}{x}\left(\ln(1+u)^{\frac{1}{u}}\right)\right\} = \frac{1}{x}\lim_{\delta x \to 0}\left\{\ln(1+u)^{\frac{1}{u}}\right\} = \frac{1}{x}\lim_{\delta x \to 0}\{\ln e\} = \frac{1}{x}$$

Given this result, we may now proof the result for the exponential function.

Theorem

$$\frac{d}{dx}e^x = e^x$$

<u>Proof</u>

$$\frac{d}{dx}\ln\left(e^x\right) = \frac{d}{dx}x\ln e = \frac{d}{dx}x = 1$$

However, by the chain rule:

$$\frac{d}{dx}\ln\left(e^x\right) = \frac{1}{e^x}\cdot\frac{d}{dx}e^x.$$

Equating both: $\frac{1}{e^x}\cdot\frac{d}{dx}e^x = 1$. Hence:

$$\frac{d}{dx}e^x = e^x.$$