The derivatives of the hyperbolic functions

The derivatives of the principle hyperbolic functions

sinh *x*

$$\frac{d}{dx}\sinh x = \cosh x$$

Proof

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\frac{1}{2}\left(e^{x} - e^{-x}\right)\\ &= \frac{1}{2}\left\{e^{x} + e^{-x}\right\}\\ &= \cosh x\end{aligned}$$

cosh *x*

$$\frac{d}{dx}\cosh x = \sinh x$$

Proof

$$\begin{aligned}\frac{d}{dx}\cosh x &= \frac{d}{dx}\left(\frac{1}{2}\left(e^{x} + e^{-x}\right)\right)\\ &= \frac{1}{2}\left(e^{x} - e^{-x}\right)\\ &= \sinh x\end{aligned}$$

tanh *x*

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

Proof

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} && [\text{By the Quotient Rule}]\\ &= \frac{1}{\cosh^2 x} && [\text{Since } \cosh^2 x - \sinh^2 x = 1]\\ &= \operatorname{sech}^2 x\end{aligned}$$

coth *x*

$$\frac{d}{dx}\coth x = -\text{cosech}^2 x$$

The proof is similar to that for tanh x

sech *x*

$$\frac{d}{dx}\text{sech } x = -\tanh x \ \text{sech} x$$

Proof

$$\begin{aligned}\frac{d}{dx}\text{sech } x &= \frac{d}{dx}\frac{1}{\cosh x} \\ &= \frac{d}{dx}(\cosh x)^{-1} \\ &= -(\cosh x)^{-2}\sinh x \\ &= -\frac{\sinh x}{\cosh x \cosh x} \\ &= -\tanh x \ \text{sech } x\end{aligned}$$

cosech *x*

$$\frac{d}{dx}\text{cosech } x = -\coth x \ \text{cosech } x$$

Problems involving derivatives of hyperbolic functions

The student should be able to apply his knowledge of differentiation and the definitions of hyperbolic functions together

Example (1)

Prove $\frac{d}{dx}\tan^{-1}(e^x) = \frac{1}{2}\text{sech} x$ [WJEC June 2003]

$$\begin{aligned}\frac{d}{dx}\tan^{-1}(e^x) &= \frac{1}{1+(e^x)^2}\times e^x \qquad \text{[By the chain rule]} \\ &= \frac{1}{\left(\frac{1+e^{2x}}{e^x}\right)} \\ &= \frac{1}{e^{-x}+e^x} \\ &= \frac{1}{2\cosh x} \qquad \text{[By the defintion of } \cosh x\text{]} \\ &= \tfrac{1}{2}\text{sech } x\end{aligned}$$

Example (2)

Find the coordinates of the minimum point of the curve $y = 3\cosh(x) - 2\sinh(x)$.

$$\frac{dy}{dx} = 3\sinh(x) - 2\cosh(x),$$

At the minimum point, $\frac{dy}{dx} = 0$, so

$$3\sinh(x) - 2\cosh(x) = 0$$
$$3\left(\frac{e^x - e^{-x}}{2}\right) - 2\left(\frac{e^x + e^{-x}}{2}\right) = 0$$
$$3\left(e^x - e^{-x}\right) - 2\left(e^x + e^{-x}\right) = 0$$
$$e^x - 5e^{-x} = 0$$
$$e^{2x} - 5 = 0$$
$$e^{2x} = 5$$
$$x = \frac{1}{2}\ln(5)$$
$$x = 0.8047 \quad (4 \text{ d.p.})$$

To check that this is a minimum point of the curve,

$$\frac{d^2y}{dx^2} = 3\cosh(x) - 2\sinh(x) = 3\cosh(0.8047) - 2\sinh(0.8047) = 2.2361 \ (4 \text{ d.p.})$$

$\frac{d^2y}{dx^2} > 0$, so there is a local minimum of the curve at this point.

We know that there are no other stationary points, so this must be the minimum point of the whole curve.

$$y = \frac{d^2y}{dx^2} = 3\cosh(0.8047) - 2\sinh(0.8047) = 2.2361 \ (4 \text{ d.p.})$$

So the coordinates of the minimum point are $(0.8047, 2.2361)$, to 4 d.p.

Integrals of hyperbolic functions

Since integration is the reverse operation of differentiation it is immediate that

$$\int \cosh x \, dx = \sinh x + c$$
$$\int \sinh x \, dx = \cosh x + c$$
$$\int \operatorname{sech}^2 x \, dx = \tanh x + c$$
$$\int \operatorname{cosech}^2 x \, dx = -\coth x + c$$
$$\int \tanh x \ \operatorname{sech} x \, dx = -\operatorname{sech} x + c$$
$$\int \coth x \ \operatorname{cosech} x \, dx = -\operatorname{cosech} x + c$$

So the student should be able to solve elementary problems involving the integration of hyperbolic functions possibly also involving hyperbolic identities.

Example (3)

Find $\int \sinh^2 x\, dx$

We start with the hyperbolic identity

$\cosh 2x = 1 + 2\sinh^2 x$

and rearrange it to obtain

$\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$

Hence

$$\int \sinh^2 x\, dx = \int \frac{1}{2}\cosh 2x - \frac{1}{2} dx$$
$$= \frac{1}{4}\sinh 2x - \frac{1}{2}x + c$$

Example (4)

Use the substitution $x = \sinh u$ to evaluate

$\int_0^2 \sqrt{1+x^2}\, dx$

We have

$x = \sinh u \quad \Rightarrow \quad dx = \cosh u \cdot du$

$\sinh u = 0 \quad \Rightarrow \quad u = 0$

$\sinh u = 2 \quad \Rightarrow \quad u = \sinh^{-1} u = 1.4436...$

Making the substitution

$$\int_0^2 \sqrt{1+x^2}\, dx = \int_0^{\sinh^{-1} 2 = 1.4436} \left(\sqrt{1+\sinh^2 u}\right)\cosh u\ du$$
$$= \int_0^{1.4436} \cosh^2 u\ du \qquad \left[\text{Since } \cosh^2 x - \sinh^2 x = 1\right]$$
$$= \frac{1}{2}\int_0^{1.4436} \cosh 2u + 1\ du \qquad \left[\text{Since } \cosh 2x = 2\cosh^2 x - 1\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}\sinh 2u + u\right]_0^{1.4436}$$
$$= \frac{1}{2}\left(\frac{1}{2}\sinh(2.8872) + 1.4436\right) - 0$$
$$= 2.96 \ \ (3 \text{ s.f.})$$