Derivatives of Inverse Trigonometric Functions

Prerequisites

In this chapter we will use the technique of implicit differentiation to find the derivatives of the inverses of the trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. The following question requires implicit differentiation.

Example (1)

Given that $y \tan x = \sec^2 y$ find $\frac{dy}{dx}$ in terms of x and y.

Solution

$$y \tan x = \sec^2 y$$
$$\frac{dy}{dx} \times \tan x + y \sec^2 x = \tan y \sec y \times \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{y \sec^2 x}{\tan y \sec y - \tan x}$$

Knowledge of the basic trigonometric identities is also a prerequisite of this chapter.

Example (2)

Rearrange $\sin^2 y + \cos^2 y = 1$ to make $\cos y$ the subject.

Solution

$$\sin^2 y + \cos^2 y = 1$$
$$\cos y = \sqrt{1 - \sin^2 y}$$

Derivatives of the inverse trigonometric functions

(1) Derivative of $\sin^{-1} x$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

Proof

Let $y = \sin^{-1} x$

Then $\sin y = x$

Differentiating this implicitly we obtain

$$\cos y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-x^2}} \qquad \text{Since } \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

(2) Derivative of $\cos^{-1} x$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

Example (3)

By imitation of the proof that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

prove $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$.

Solution

Let $y = \cos^{-1} x$

Then $\cos y = x$

$$\therefore -\sin y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$= -\frac{1}{\sqrt{1-x^2}} \qquad \text{Since } \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$$

(3) Derivative of $\tan^{-1} x$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

Proof

Let $y = \tan^{-1} x$

Then $\tan y = x$

$$\sec^2 y \times \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{dy}{dx} = \frac{1}{1+x^2}$$ $\tan^2 y + 1 = \sec^2 y \Rightarrow \sec^2 y = 1 + x^2$

Applications of the chain rule

You should be able to find the derivatives of expressions such as $\sin^{-1}(3x)$ that are composite functions.

Example (4)

Differentiate $y = \sin^{-1}(3x)$.

Solution

$y = \sin^{-1}(3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \times 3$$ By the chain rule

$$= \frac{3}{\sqrt{1-9x^2}}$$

Problems involving derivatives of inverse trigonometric functions

Since we have added the derivatives of the inverse trigonometric functions to your store of derivatives, you are expected to be able to differentiate them in the context of a regular problem.

Example (5)

A function is defined by the rule

$f(x) = \cos^{-1} x + \sqrt{1-x^2} + x$

(*a*) What is the maximal domain of *f*?

(*b*) Show that *f* has a stationary value at $x = 0$.

(*c*) Explain why *f* does not have a stationary value at $x = -1$.

Solution

(*a*) $f(x) = \cos^{-1} x + \sqrt{1-x^2} + x$ is the sum of three functions.

The function $y = x$ is defined for all *x*.

The function $y = \cos^{-1} x$ is defined only on the interval $-1 \le x \le 1$.

The function $y = \sqrt{1-x^2}$ is undefined for when $1 - x^2 < 0$. That is for $|x| < 1$. That is it is defined only for $-1 \le x \le 1$. The maximal domain (domain of largest size) is $-1 \le x \le 1$ or $[-1,1]$.

(*b*)
$$f(x) = \cos^{-1} x + \sqrt{1-x^2} + x$$
$$= \cos^{-1} x + \left(1-x^2\right)^{\frac{1}{2}} + x$$

$$f'(x) = -\frac{1}{\sqrt{1-x^2}} + \frac{1}{2} \times \left(1-x^2\right)^{-\frac{1}{2}} \times -2x + 1$$
$$= -\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} + 1$$

For turning points $f'(x) = 0$

$$-\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} + 1 = 0 \qquad \text{undefined if } |x| = 1$$
$$\frac{1+x}{\sqrt{1-x^2}} = 1$$
$$1 + x = \sqrt{1-x^2}$$
$$(1+x)^2 = 1 - x^2$$
$$1 + 2x + x^2 = 1 - x^2$$
$$2x^2 + 2x = 0$$
$$x(x+1) = 0$$
$$x = 0 \text{ or } x = -1 \qquad \text{but } f'(x) \text{ is undefined at } x = -1$$

So there is a turning point (stationary value) at $x = 0$.

(*c*) At $x = -1$ the graph of $f(x) = \cos^{-1} x + \sqrt{1-x^2} + x$ comes to an abrupt halt. The point $x = -1$ is included in the domain but *f* is undefined for $x < -1$. Therefore *f* is not continuous at $x = -1$. The derivative of *f* is

$f'(x) = 1 - \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$

This is undefined at $x = -1$ since on substitution of $x = -1$ into it we are required to divide by zero, which is not possible.

A sketch of the graph of *f* is