Derivatives of sine and cosine

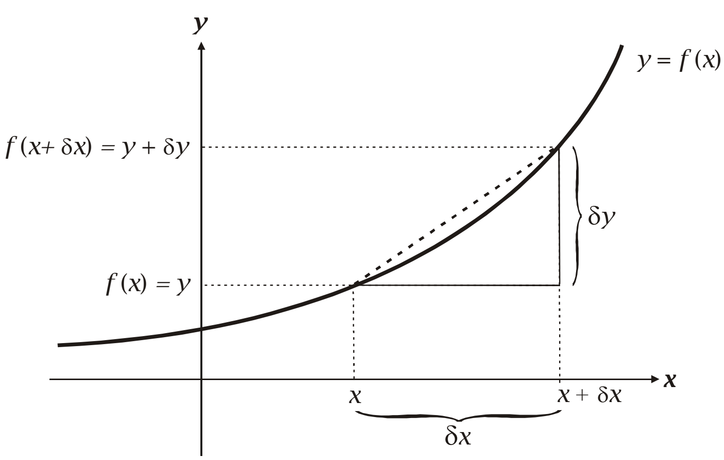
Introduction

The purpose of this chapter is to prove the following results for the derivatives of the trigonometric functions $\sin x$ and $\cos x$.

$\frac{d}{dx}\sin x = \cos x$ $\qquad$ $\frac{d}{dx}\cos x = -\sin x$

Prerequisites

You should understand differentiation from first principles. If $y = f(x)$ is a function then its derivative is $f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{\delta y}{\delta x}\right\} = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$ where δx stands for a small increment in *x*, as the following diagram shows.



In the symbol δx the δ cannot be separated from the *x*. On its own it is meaningless. It may be read as "small change in". Often the symbol h instead of δx when the formula becomes

$$\frac{dy}{dx} = \lim_{h \to 0}\left\{\frac{\delta y}{h}\right\} = \lim_{h \to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\}$$

Example (1)

Use the formula

$$\frac{dy}{dx} = \lim_{h\to 0}\left\{\frac{\delta y}{h}\right\} = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$$

to find the derivative of $y = x^3 + x$ from first principles.

Solution

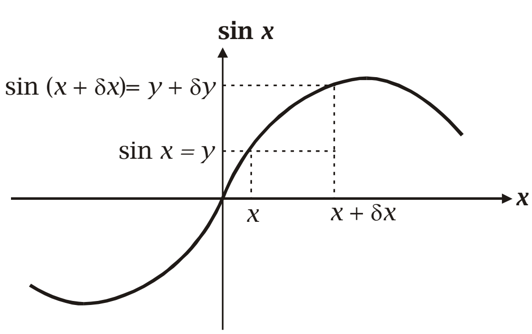
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{(x+h)^3+(x+h)-(x^3+x)}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{x^3+3x^2h+3xh^2+h^3+(x+h)-(x^3+x)}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{3x^2h+3xh^2+h^3+h}{h}\right\}\\ &= \lim_{h\to 0}\left\{3x^2+3xh+h^2+1\right\}\\ &= 3x^2+1\end{aligned}$$

Proof that the derivative of sin*x* is cos*x*

To prove $\frac{d}{dx}\sin x = \cos x$

Proof

Let $y = \sin x$ be a value of $f(x) = \sin x$ for arbitrary *x*, let δx be a small increase in the value of *x* and δy be the corresponding small increase in the value of *y*, so that $y + \delta y = f(x+\delta x) = \sin(x+\delta x)$.



The proof that $\frac{d}{dx}\sin x=\cos x$ requires the following statements

(1) $\sin(x+\delta x)-\sin x=2\cos\left(x+\tfrac{1}{2}\delta x\right)\sin\left(\tfrac{1}{2}\delta x\right)$

(2) $\lim_{\delta x\to 0}\cos(x+\delta x)=\cos x$

(3) $\lim_{\delta x\to 0}\left(\frac{\sin x}{x}\right)=1$

We have placed proofs of these into an appendix at the end of the chapter and will assume them here.[1] The formula for differentiation from first principles is given by $\frac{d}{dx}f(x)=\lim_{\delta x\to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\}$. Therefore

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\delta x\to 0}\left\{\frac{\sin(x+\delta x)-\sin(x)}{\delta x}\right\}\\
&= \lim_{\delta x\to 0}\left\{\frac{2\cos\left(x+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)}{\delta x}\right\} && \text{by (1)}\\
&= \lim_{\delta x\to 0}\left\{\frac{\cos\left(x+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x}\right\}\\
&= \lim_{\delta x\to 0}\left\{\cos\left(x+\frac{1}{2}\delta x\right)\right\}\times\lim_{\delta x\to 0}\left\{\frac{\sin\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x}\right\}\\
&= \cos x && \text{by (2) and (3)}
\end{aligned}
$$

Example (3)

Show that $\frac{d}{dx}\sin(x+c)=\cos(x+c)$ where c is a real number.

Solution

[1] In the proof that follows we also tacitly assume other results about limits that strictly should be proven.

$$\frac{d}{dx}\sin(x+c) = \lim_{\delta x\to 0}\left\{\frac{\sin(x+c+\delta x)-\sin(x+c)}{\delta x}\right\}$$
$$= \lim_{\delta x\to 0}\left\{\frac{2\cos\left(x+c+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)}{\delta x}\right\}$$
$$= \lim_{\delta x\to 0}\left\{\frac{\cos\left(x+c+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x}\right\}$$
$$= \lim_{\delta x\to 0}\left\{\cos\left(x+c+\frac{1}{2}\delta x\right)\right\}\times\lim_{\delta x\to 0}\left\{\frac{\sin\left(\frac{1}{2}\delta x\right)}{\frac{1}{2}\delta x}\right\} = \cos(x+c)$$

To prove $\frac{d}{dx}\cos x = -\sin x$

Proof

We use the fact that

$\cos x \equiv \sin(x+90°)$

Then

$$\frac{d}{dx}\cos x = \frac{d}{dx}\left(\sin(x+90°)\right)$$
$$= \cos(x+90°) \qquad (*)$$
$$= \sin(x+180°)$$
$$= -\sin x$$

The line marked $(*)$ in this follows from the result we established in example (3) that

$$\frac{d}{dx}\sin(x+c) = \cos(x+c)$$

Appendix

In the proof that $\frac{d}{dx}\sin x = \cos x$ we required the following statements

(1) $\sin(x+\delta x) - \sin x = 2\cos\left(x+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)$

(2) $\lim_{\delta x\to 0}\cos(x+\delta x) = \cos x$

(3) $\lim_{\delta x\to 0}\left(\frac{\sin x}{x}\right) = 1$

We will now prove these.

(1) $\sin(x+\delta x)-\sin x = 2\cos\left(x+\frac{1}{2}\delta x\right)\sin\left(\frac{1}{2}\delta x\right)$

Proof

This also requires prior knowledge of the following compound angle formulae

$(1')$ $\cos(A+B)=\cos A\cos B-\sin A\sin B$

$(2')$ $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$(3')$ $\cos 2A = 1-2\sin^2 \text{A}$

$(4')$ $\sin 2A = 2\sin A\cos A$

$$
\begin{aligned}
\text{RHS} &= 2\cos\left(x+\tfrac{1}{2}\delta x\right)\sin\left(\tfrac{1}{2}\delta x\right) \\
&= 2\left(\cos x\cos\left(\tfrac{1}{2}\delta x\right)-\sin x\sin\left(\tfrac{1}{2}\delta x\right)\right)\sin\left(\tfrac{1}{2}\delta x\right) && \text{By } (1') \\
&= 2\cos x\cos\left(\tfrac{1}{2}\delta x\right)\sin\left(\tfrac{1}{2}\delta x\right)-2\sin x\sin^2\left(\tfrac{1}{2}\delta x\right) \\
&= \cos x\sin\delta x-\sin x\left(1-\cos\delta x\right) && \text{By } (3') \text{ and } (4') \\
&= \cos x\sin\delta x-\sin x+\cos\delta x\sin x \\
&= \sin(x+\delta x)-\sin x && \text{By } (2') \\
&= \text{LHS}
\end{aligned}
$$

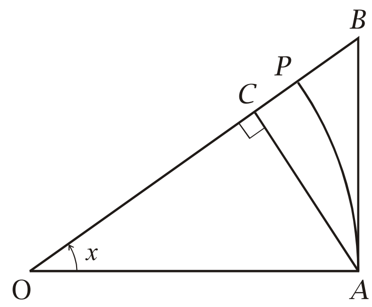
(2) $\lim_{\delta x\to 0}\cos(x+\delta x)=\cos x$

Proof

By substituting $\delta x = 0$ into the left-hand side of this equation we simply get $\cos x = \cos x$ which demonstrates the result. However, this is an informal proof of a result that is intuitively obvious. A rigorous or formal proof would require the development of further theory.

(3) To prove $\lim_{\delta x\to 0}\left(\frac{\sin x}{x}\right)=1$

Proof



In the right-angled triangle OAB let OA be of unit length, and the angle $\angle BOA$ be x radians, where x is small. Let AC be perpendicular to OB, and AP be an arc of a circle. Then by geometric intuition

$$AC < \text{arc } AP < AB$$

But $AC = \sin x$, arc $AP = x$, $AB = OB \sin x$

Hence

$$\sin x < x < OB \sin x$$

$$1 < \frac{x}{\sin x} < OB$$

As $x \to 0$, OB tends to equality with OA, hence

$$\lim_{x \to 0} OB = 1$$

We have

$$1 < \lim_{x \to 0} \frac{x}{\sin x} < \lim_{x \to 0} OB$$

That is

$$1 < \lim_{x \to 0} \frac{x}{\sin x} < 1$$

Hence

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$