Determinant and Inverse of a 2×2 matrix

Prerequisites

You should be familiar with the addition and multiplication of matrices.

Definition of the determinant of a 2 ×2 matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the number $ad - bc$ is called its *determinant.* It is denoted by

$$\det A \text{ or } \begin{vmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{vmatrix}$$

In the second of these symbols the straight lines indicate that we have a determinant rather than a matrix.

Example (1)

If $A = \begin{pmatrix} -3 & -2 \\ 1 & 6 \end{pmatrix}$ find $\det A$

Solution

$$\det A = \begin{vmatrix} 6 & 2 \\ 1 & -3 \end{vmatrix} = (-3 \times 6) - (-2 \times 1) = -16$$

Note that a determinant is a number; it is not a matrix. The operation of taking a determinant is a mapping from a square matrix to a number.

Example (2)

Find the determinant of $A = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$

Solution

$$\det A = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix} = (-1 \times -2) - (2 \times 3) = -4$$

Inverse of a 2 ×2 matrix

The inverse of a matrix, A, is the matrix which, when multiplied by A gives the identity matrix I. From this definition, if A^{-1} represents the inverse of A then

$$AA^{-1} = A^{-1}A = \mathbf{I}$$

Formula for the inverse of a matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This demonstrates the importance of determinants.

Example (3)

Let $A = \begin{pmatrix} -3 & -2 \\ 1 & 6 \end{pmatrix}$ find A^{-1} and verify that $AA^{-1} = \mathbf{I}$

Solution

$$A^{-1} = \frac{1}{(-3\times 6)-(-2\times 1)}\begin{pmatrix} 6 & 2 \\ -1 & -3 \end{pmatrix}$$

$$= -\frac{1}{16}\begin{pmatrix} 6 & 2 \\ -1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -6/16 & -2/16 \\ 1/16 & 3/16 \end{pmatrix}$$

To verify that $AA^{-1} = I$

$$AA^{-1} = \begin{pmatrix} -3 & -2 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -6/16 & -2/16 \\ 1/16 & 3/16 \end{pmatrix}$$

$$= \begin{pmatrix} \left(-3\times -\frac{6}{16}\right)+\left(-2\times\frac{1}{16}\right) & \left(-3\times -\frac{2}{16}\right)+\left(-2\times\frac{3}{16}\right) \\ \left(1\times -\frac{6}{16}\right)+\left(6\times\frac{6}{16}\right) & \left(1\times -\frac{2}{16}\right)+\left(6\times\frac{3}{16}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{18-2}{16} & \frac{-6+6}{16} \\ \frac{-6+6}{16} & \frac{-2+18}{16} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathbf{I}$$

Proof of inverse formula

To prove that the inverse of A is $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

then

$$\begin{aligned}
A^{-1}\ A &= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{1}{ad - bc}\begin{pmatrix} ad - bc & bd - bd \\ -ca + ac & -cb + ad \end{pmatrix} \\
&= \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \frac{ad - bc}{ad - bc}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathbf{I}
\end{aligned}$$

Hence, $AA^{-1} = \mathbf{I}$ and A^{-1} is the inverse of A.

An important point to note is that *not all matrices have an inverse.* Matrices whose determinant is 0 do not have an inverse. They are called *signular* matrices.

Dividing by a matrix

Division is the inverse operation of multiplication. Consider a number *x*., then the inverse of *x* under the operation of multiplication is called its *reciprocal.* Provided $x \neq 0$ the inverse is

$$x^{-1} = \frac{1}{x}$$

When a number is multiplied by its reciprocal (inverse) the result is the identity.

$$x \times x^{-1} = x \times \frac{1}{x} = 1$$

To divide a number *y* by another *x* is the same as multiplying *y* by the reciprocal of *x*.

$$y \div x = y \times x^{-1} = y \times \frac{1}{x}$$

So, in the same way, division of a matrix *B* by another matrix *A* is equivalent to multiplication of *B* by A^{-1}, the inverse of *A*.. However, when dividing one matrix by another care must be taken, because matrix multiplication is not commutative; that is, generally $AB \neq BA$. So we do not use the symbol $\div$ for matrices, because it is ambiguous.

Example (4)

Let $A = \begin{pmatrix} -3 & -2 \\ 1 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$. (a) Find X such that $XA = B$; (b) Find Y such that $AY = B$

Solution

In example (3) we showed that

$$A^{-1} = \begin{pmatrix} -6/16 & -2/16 \\ 1/16 & 3/16 \end{pmatrix}$$

then

$$XA = B$$
$$XAA^{-1} = BA^{-1}$$
$$X\mathbf{I} = BA^{-1}$$
$$X = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -6/16 & -2/16 \\ 1/16 & 3/16 \end{pmatrix} = \begin{pmatrix} 8/16 & 8/16 \\ 1/16 & 3/16 \end{pmatrix}$$
$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$\mathbf{I}X = A^{-1}B$$
$$X = \begin{pmatrix} -6/16 & -2/16 \\ 1/16 & 3/16 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6/16 & -14/16 \\ -1/16 & 5/16 \end{pmatrix}$$

When it comes to division in matrix algebra, this is equivalent to multiplication by an inverse, and care must be taken whether the multiplication is on the left or the right. The identity

$$A^{-1}A = \mathbf{I}$$

is used to clarify this. As commutativity fails for matrix multiplication in general

$$A^{-1}B \neq BA^{-1}.$$

Inverse matrices and matrix multiplication

We can prove the following relationship between inverses matrices and matrix multiplication

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Example (5)

For $A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix}$

verify the relationship $(AB)^{-1} = B^{-1}A^{-1}$

Solution

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} \quad B^{-1} = -\frac{1}{2}\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 8 & -9 \end{pmatrix}$$

$$(AB)^{-1} = -\frac{1}{4}\begin{pmatrix} -9 & -5 \\ -8 & -4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 9 & 5 \\ 8 & 4 \end{pmatrix}$$

$$\begin{aligned} B^{-1}A^{-1} &= -\frac{1}{2}\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} \\ &= -\frac{1}{4}\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} \\ &= -\frac{1}{4}\begin{pmatrix} -9 & -5 \\ -8 & -4 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 9 & 5 \\ 8 & 4 \end{pmatrix} \end{aligned}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$

Example (6)

Prove the formula $(AB)^{-1} = B^{-1}A^{-1}$

Solution

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

then

$$AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{\det AB}\begin{pmatrix} a_3b_2 + a_4b_4 & -\left(a_1b_2 + a_2b_4\right) \\ -\left(a_3b_1 + a_4b_3\right) & a_1b_2 + a_2b_3 \end{pmatrix}$$

where

$$\begin{aligned} \det(AB) &= \left(a_1b_1 + a_2b_3\right)\left(a_3b_2 + a_4b_4\right) - \left(a_1b_2 + a_2b_4\right)\left(a_3b_1 + a_4b_3\right) \\ &= a_1a_3b_1b_2 + a_1a_4b_1b_4 + a_2a_3b_2b_3 + a_2a_4b_3b_4 - a_1a_3b_1b_2 - a_1a_4b_2b_3 - a_2a_3b_1b_4 - a_2a_4b_3b_4 \\ &= a_1a_4\left(b_1b_4 - b_2b_3\right) + a_2a_3\left(b_2b_3 - b_1b_4\right) \\ &= a_2a_3\left(b_1b_4 - b_2b_3\right) - a_1a_4\left(b_2b_3 - b_1b_4\right) \\ &= \left(a_2a - a_1a_4\right)\left(b_2b_3 - b_1b_4\right) \end{aligned}$$

Further

$$A^{-1} = \frac{1}{a_1a_4 - a_2a_3}\begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{b_1b_4 - b_2b_3}\begin{pmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{pmatrix}$$

$$
\begin{aligned}
B^{-1}A^{-1} &= \frac{1}{(a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3)}\begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}\begin{pmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{pmatrix} \\
&= \frac{1}{(a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3)}\begin{pmatrix} a_4b_4 + a_2b_3 & -a_4b_2 - a_2b_3 \\ -a_3b_4 - a_1b_3 & a_3b_2 + a_1b_4 \end{pmatrix} \\
&= (AB)^{-1}
\end{aligned}
$$