Determinants

Prerequisites

You should be familiar with the definition of a determinant for a 2×2 matrix. To remind you,, for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the number $ad - bc$ is called its *determinant.* It is denoted by

$$\det A \text{ or } \begin{vmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{vmatrix}$$

The purpose of this chapter is to extend the definition of a determinant to include 3×3 matrices, and to look at certain relationships that exist between determinants.

Determinant of a 3 × 3 matrix

The determinant of a 3×3 matrix is defined in terms of 2×2 determinants

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

then

$$\begin{aligned} \det A &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \end{aligned}$$

Example (1)

Find the determinant of $\begin{pmatrix} -1 & 2 & 2 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$

Solution

$$\begin{aligned} \det A &= -1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\ &= -1(2+1) + 2(-1-6) + 2\,(3-1) \\ &= (-1 \times 3) + (2 \times -7) + (2 \times 2) \\ &= -3 - 14 + 4 \\ &= -13 \end{aligned}$$

The definition seems rather cumbersome at first, but in practice can be quite easy to recall. Thus, in the above example we begin by taking the first entry in the first row and multiplying it by the determinant of the matrix diagonally below it.

$$\begin{pmatrix} -1 & 2 & 2 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

So

$$\det A = -1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + \ldots$$

Now proceed to the next column. We are dealing with the contribution to the determinant from the "2" in the second column. But first…

$$\begin{pmatrix} -1 & 2 & 2 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}\begin{matrix} \\ 3 \\ 1 \end{matrix}$$

… in the mind's eye, move the column as indicated. Then, multiply the second entry of the first row by the determinant of the square 2×2 matrix just formed.

$$\begin{pmatrix} -1 & 2 & 2 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}\begin{matrix} \\ 3 \\ 1 \end{matrix}$$

So

$$\det A = -1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + \ldots$$

Now proceed to the last column – in this example the "2" in the third column.

$$\begin{pmatrix} -1 & 2 & 2 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}\begin{matrix} & \\ 3 & 1 \\ 1 & 1 \end{matrix}$$

Using your imagination move the array of numbers as indicated. Then, multiply the third entry of the first row by the determinant of square 2×2 matrix just formed.

$$\begin{pmatrix} -1 & 2 & 2 \\ & & -1 \\ & & 2 \end{pmatrix} \rightarrow \boxed{\begin{matrix} 3 & 1 \\ 1 & 1 \end{matrix}}$$

And this gives the determinant as

$$\det A = -1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

All that remains to be done is to work out the individual determinants and add up the sum

$$\begin{aligned} \det A &= -1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\ &= -1(2+1) + 2(-1-6) + 2\,(3-1) \\ &= (-1\times 3) + (2\times -7) + (2\times 2) \\ &= -3 - 14 + 4 \\ &= -13 \end{aligned}$$

Example (2)

Find $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 1 & -6 \\ -1 & -3 & 2 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} 2 & 3 & 4 \\ 5 & 1 & -6 \\ -1 & -3 & 2 \end{vmatrix} &= 2\begin{vmatrix} 1 & -6 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} -6 & 5 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} 5 & 1 \\ -1 & -3 \end{vmatrix} \\ &= 2(-16) + 3\,(-4) + 4\,(-14) \\ &= -32 - 12 - 56 \\ &= -100 \end{aligned}$$

Relationships between determinants

It is possible to prove certain relationships between determinants.

Example (3)

For a 2×2 matrix, if A^{-1} is the inverse of A, prove: $\det A = \dfrac{1}{\det\left(A^{-1}\right)}$

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det A = ad - bc$

We have already shown that the inverse of a 2×2 matrix is $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Hence $\det A^{-1}=\dfrac{1}{\left(\det A^2\right)}\left(ad-bc\right)=\dfrac{1}{\left(\det A\right)^2}\times\det A=\dfrac{1}{\det A}$

Example (4)

Prove for two 2×2 matrices A, B

$\det\left(AB\right)=\det A\det B$

Solution

Let $\quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

then

$$\begin{aligned}
&\det A=ad-bc\\
&\det B=ps-qr\\
&\det A\ \det B=\left(ad-bc\right)\left(ps-qr\right)=adps+bcqr-adqr-bcps
\end{aligned}$$

$$AB=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

$$\begin{aligned}
\det\left(AB\right)&=\begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}\\
&=\left(ap+br\right)\left(cq+ds\right)-\left(aq+bs\right)\left(cp+dr\right)\\
&=acpq+adps+bcqr+bdrs-acpq-adqr-bcps-bdrs\\
&=adps+bcqr-adqr-bcps\\
&=\det A\det B
\end{aligned}$$

To prove the following two formulae we would require mathematical induction, so we just state the results here.

(1) For for any n and any $m\times m$ square matrices $A_1A_2....A_n$
$\det\left(A_1A_2....A_n\right)=\det A_1\det A_2....\det A_n$

(2) For any $n\times n$ matrix A, and any scalar k
$\det\left(k^nA\right)=k^n\det A$

Example (5)

The transpose of a 2×2 $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the matrix $A^T=\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$. The rows are written as columns and vice-versa. Prove for any 2×2 that

$\det A=\det A^T$

Solution

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det A^T = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{22}a_{12} = \det A$$