Difference of two sample means

The z-test

Suppose we have two independent samples drawn one from each of two normally distributed populations of differing known variance. That is, we are given that

$$X_1 \sim N\left(\mu_1, \sigma_1{}^2\right)$$
$$X_2 \sim N\left(\mu_2, \sigma_2{}^2\right)$$

where σ_1^2 and σ_2^2 are known. We are asked to test whether the two population means are equal at a given significance level. To say that the samples are independent is to say that values of X are not in any way associated or paired with values of X_2.

$$\mathrm{H}_0 : \mu_1 = \mu_2$$
$$\mathrm{H}_1 : \mu_1 \neq \mu_2 \qquad \text{(two tailed)}$$

or if H_1 is either $\mu_1 < \mu_2$ or $\mu_1 > \mu_2$ (one tailed)

To carry out this test we test the difference of the means, for it turns out that

$$\bar{D} = \bar{X}_1 - \bar{X}_2$$

is normally distributed if both X_1 and X_2 are normally distributed.

$$\bar{D} = \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}\right)$$

Hence the test statistic is:

$$Z_{test} = \frac{\left|\bar{X}_1 - \bar{X}_2\right|}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

The critical value is found using the tables for the standardised normal distribution and finding the z value corresponding to the given significance level.

The first example illustrates the case where the population variances are known and different.

Example (1)

A random sample is taken from each of two normally distributed populations. The sample size of the first population is 10 with sample mean 18.2. The sample size of the second population is 12, with sample mean 15.3. It is known that the population variances are 2.5 and 2.8 respectively. Test at the 1% level whether the means of the two populations are equal.

First sample

$X_1 \sim N(\mu_1, \sigma_1^2)$
$n_1 = 10,\ \bar{X}_1 = 18.2,\ \sigma_1^2 = 2.5$

Second sample
$X_2 \sim N(\mu_2, \sigma_2^2)$
$n_2 = 12,\ \bar{X}_2 = 15.3,\ \sigma_2^2 = 2.8$

$H_0 : \mu_1 = \mu_2$
$H_1 : \mu_1 \neq \mu_2$ two-tailed significance : $\alpha = 1\%$

$$Z_{test} = \frac{\left|\bar{X}_1 - \bar{X}_2\right|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$
$$= \frac{\left|18.2 - 15.3\right|}{\sqrt{\frac{2.5}{10} + \frac{2.8}{12}}}$$
$$= 4.17\ (3\ s.f.)$$

$Z_{\text{critical}} = P(Z > 0.995)$ since α=1% and the test is 2-tailed
$= 2.576$

$Z_{\text{test}} > Z_{\text{critical}}$
$\therefore$ Accept H_1, reject H_0
The two populations are not equal.

When it is given that the population variances are identical, then the test becomes even simpler, since then

$$Z_{\text{test}} = \frac{\left|\bar{X}_1 - \bar{X}_2\right|}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where σ^2 is the common variance.

Example (2)

Two independent random samples were taken, one from each of two normally distributed populations both of which may be assumed to have the same variance, σ=16. If the first sample had size 8 and mean 4 and the second have size 24 and mean 5, test the null hypothesis $\mu_1 - \mu_2 = 0$ against the alternative hypothesis $\mu_1 - \mu_2 \neq 0$ at the 1% significance level, where μ_1 and μ_2 are the means of the respective populations.

$X_1 \sim N(\mu_1, \sigma_1^2)$ $\quad$ $n_1 = 8,\ \bar{X}_1 = 4$

$X_2 \sim N(\mu_2, \sigma_2^2)$ $\quad$ $n_2 = 24,\ \bar{X}_2 = 5$

$\sigma^2 = 16$ $\quad$ $\sigma = \sqrt{16} = 4$

$H_0 :\ \mu_1 - \mu_2 = 0$

$H_1 :\ \mu_1 - \mu_2 \neq 0$ $\quad$ two-tailed $\quad$ $\alpha = 1\%$

$$\begin{aligned} Z_{test} &= \frac{\left|\bar{X}_1 - \bar{X}_2\right|}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ &= \frac{|4 - 5|}{4\sqrt{\frac{1}{8} + \frac{1}{24}}} \\ &= \frac{1}{4\sqrt{\frac{3+1}{24}}} \\ &= \frac{1}{4\sqrt{\frac{1}{6}}} \\ &= \frac{\sqrt{6}}{4} = 0.612\ (3\ s.f.) \end{aligned}$$

$$Z_{critical} = P(Z < 0.995) = -2.576$$
$$Z_{test} < Z_{critical}$$

$\therefore$ Accept H_0, reject H_1

The two means are the same.

The t-test for unrelated data

There is also a test for the difference of two sample means based on Students' *t* distribution. This is very closely related to the *z*-test described above, and not appreciably different from it. Therefore, use the *z*-test.