Differentiation of Products, Chains and Quotients

Prerequisites

You should be aware of the following rules for differentiation

(**1**) Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} \qquad \text{where } h \text{ is a small increase in } x$$

(**2**) Differentiation of basic polynomial functions of the form $f(x) = x^n$

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(**3**) The rule for constant multiples

$$(af(x))' = af'(x)$$

$$\frac{d}{dx}(ay) = a\frac{dy}{dx}$$

(**4**) The rule for sums of functions

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(**5**) Derivatives of exponential and natural logarithm

$$\frac{d}{dx}e^x = e^x$$

(**6**) Derivatives of sine and cosine

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

You should also be familiar with the following definitions

(**7**) $\sec x = \dfrac{1}{\cos x}$ $\quad$ $\operatorname{cosec} x = \dfrac{1}{\sin x}$ $\quad$ $\cot x = \dfrac{1}{\tan x}$

And the following trigonometric identities

(8) $\sin^2 x + \cos^2 \equiv 1$ $\quad$ $\tan x \equiv \dfrac{\cos x}{\sin x}$

The product rule

A product of functions is simply when two different functions of x are multiplied. For example, in the expression $(2x^2 - 3x + 1)(2x - 3)$ there are two functions multiplied together. Writing the first function as $f(x) = (2x^2 - 3x + 1)$, the second function as $g(x) = (2x - 3)$ their product is

$f(x)g(x) = (f \times g)(x) = (2x^2 - 3x + 1)(2x - 3)$

Our concern here is with the differentiation of a product of functions. This is given by

$$(f \times g)' = f' \times g + f \times g' \qquad \frac{d}{dx}(u \times v) = \frac{du}{dx} \times v + \frac{dv}{dx} \times u$$

These are just two different ways of expressing the *same* rule. In words, the rule is, "There are two functions multiplied together and you want the derivative of the product. To find this, differentiate the first function and multiply this result by the second function. To this, add the derivative of the second function multiplied by the first function."

Example (1)

Differentiate $(2x^2 - 3x + 1)(2x - 3)$

Solution

$h(x) = f(x) \times g(x)$

where $f(x) = (2x^2 - 3x + 1)$, and $g(x) = (2x - 3)$

$$\begin{aligned} f(x) &= (2x^2 - 3x + 1) \\ f'(x) &= 4x - 3 \\ g(x) &= (2x - 3) \\ g'(x) &= 2 \end{aligned}$$

The rule is

$$h' = (f \times g)' = f' \times g + f \times g'$$

On substitution into this we get

$$\begin{aligned} h'(x) &= (4x - 3)(2x - 3) + (2x^2 - 3x + 1) \times 2 \\ &= 8x^2 - 18x + 9 + 4x^2 - 6x + 2 \\ &= 12x^2 - 24x + 11 \end{aligned}$$

Remark

Example (1) could be solved by first expanding $(2x^2 - 3x + 1)(2x - 3)$ and then differentiating. However, the aim here is to practice using the rule $(f \times g)' = f' \times g + f \times g'$ so we solve it by differentiating first then collecting terms.

Example (2)

Differentiate $h(x) = \cos x \sin x$

Solution

$h(x) = f(x) \times g(x)$

where $f(x) = \cos x,$ and $g(x) = \sin x$

The rule is

$$h' = (f \times g)' = f' \times g + f \times g'$$

$$f(x) = \cos x$$
$$f'(x) = -\sin x$$
$$g(x) = \sin x$$
$$g'(x) = \cos x$$

$$\begin{aligned} h' &= (f \times g)' \\ &= f' \times g + f \times g' \\ &= -\sin x \times \sin x + \cos x \times \cos x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

Example (3)

Using the rule for the differentiation of products as

$$\frac{d}{dx}(u \times v) = \frac{du}{dx} \times v + \frac{dv}{dx} \times u$$

differentiate $y = e^x \sin x$

Solution

$y(x) = u(x)v(x)$

where $u(x) = e^x,$ and $v(x) = \sin x$

The rule for differentiating a product is

$$\frac{d}{dx}(u \times v) = \frac{du}{dx} \times v + \frac{dv}{dx} \times u$$

Here $\quad u = e^x \qquad \frac{du}{dx} = e^x$

$\qquad\quad v = \sin x \qquad \frac{dv}{dx} = \cos x$

Hence

$$\begin{aligned} \frac{d}{dx}(u \times v) &= \frac{du}{dx} \times v + \frac{dv}{dx} \times u \\ &= e^x \sin x + \cos x \times e^x \\ &= e^x (\sin x + \cos x) \end{aligned}$$

The chain rule

A composition of functions is when we apply a second function to the result of a first function. If f and g are functions then fg is the composite function: g followed by f, with the rule

$fg(x) = f(g(x))$

In this we follow the convention that the expression fg means g followed by f, indicating the composition of functions and **not** g multiplied by f, which is a product of functions. It is vital to keep a distinction between the product of functions and the composition of functions.

Example (4)

Write $h(x) = \sin^3 x$ as the composition of two functions.

Solution

$h(x) = \sin^3 x$ is a composite function which says "take x, apply the sine function to it, then apply the cube function on the result". Let $f(x) = x^3$ and $g(x) = \sin x$, then

$$\begin{aligned} h(x) &= fg(x) \\ &= f(g(x)) \\ &= f(\sin x) \\ &= (\sin x)^3 \\ &= \sin^3 x \end{aligned}$$

Another name for the composition of two functions is a *chain*. The rule for differentiating composite functions is called the *chain rule.*

$(fg)' = (f'g) \times g'$

$\frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$ where $u(x) = u(v(x))$

Looking at the form $fg(x) = f(g(x))$ for a composite function, then let us call $g(x)$ the "inside" function and $f(x)$ the "outside" function. With this idea the chain rule can be remembered as, "Differentiate the outside function and apply it to the inside function, then multiply the result by the derivative of the inside function."

Example (5)

Find the derivative of $h(x) = \sin^3 x$

Solution

$h(x) = \sin^3 x = (\sin x)^3$

We saw in example (3) that this may be written as $fg(x) = f(g(x))$ where

$g(x) = \sin x$ and $f(x) = x^3$

The "outside" function is $f(x) = x^3$; its derivative is $f'(x) = x^3$. When applied to the inside function, which is $g(x) = \sin x$, this gives $f'(g(x)) = 3(\sin x)^2 = 3\sin^2 x$. The derivative of the inside function is $\cos x$.

$$f(x) = x^3 \qquad f'(x) = 3x^2$$
$$g(x) = \sin x \qquad g'(x) = \cos x$$
$$f'g(x) = f'(g(x)) = f'(\sin x) = 3(\sin x)^2 = 3\sin^2 x$$
$$(fg)'(x) = f'(g(x)) \times g'(x) = 3\sin^2 x \cos x$$

Example (6)

Differentiate $\sin\left(\frac{x}{3}\right)$

Solution

$\sin\left(\frac{x}{3}\right)$ is a composite of functions

$\sin\left(\frac{x}{3}\right) = f(g(x))$ where $f(x) = \sin x$ and $g(x) = \frac{x}{3}$

$f'(x) = \cos x$

$g'(x) = \frac{1}{3}$

Applying the chain rule

$$\begin{aligned}\sin\left(\frac{x}{3}\right)' &= f'(g(x)) \times g'(x) \\ &= \left(\cos\frac{x}{3}\right) \times \frac{1}{3} \\ &= \frac{1}{3}\cos\left(\frac{x}{3}\right)\end{aligned}$$

Example (7)

Differentiate $e^{-3x} + \ln(x+5)$

Solution

Both e^{-3x} and $\ln(x+5)$ are composite functions.

(*i*) In the case of e^{-3x} let

$$f(x) = e^x \text{ and } g(x) = -3x$$

$$f'(x) = e^x \qquad f'(g(x)) = e^{-3x} \qquad g'(x) = -3$$

$$\frac{d}{dx}e^{-3x} = f'(g(x)) \times g'(x) = -3e^{-3x}$$

(*ii*) In the case of $\ln(x+5)$

$$f(x) = \ln x \text{ and } g(x) = x+5$$

$$f'(x) = \frac{1}{x} \qquad f'(g(x)) = \frac{1}{x+5} \qquad g'(x) = 1$$

$$\frac{d}{dx}\ln(x+5) = f'(g(x)) \times g'(x) = \frac{1}{x+5}$$

Overall

$$\frac{d}{dx}\left(e^{-3x} + \ln(x+5)\right) = -3e^{-3x} + \frac{1}{x+5}$$

The quotient rule

This is the rule for differentiating one function divided by another.

$$\left(\frac{f}{g}\right)' = \frac{f' \times g - f \times g'}{g^2}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

Example (8)

Differentiate $\frac{x}{\sin x}$

Solution

$f(x) = x$

$f'(x) = 1$

$g(x) = \sin x$

$g'(x) = \cos x$

$$\left(\frac{f}{g}\right)' = \frac{f' \times g - f \times g'}{g^2}$$
$$= \frac{1 \times \sin x - x \times \cos x}{\sin^2 x}$$
$$= \frac{\sin x - x \cos x}{\sin^2 x}$$

Example (9)

Use the formula

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

to differentiate $\tan x = \frac{\sin x}{\cos x}$

Solution

$u(x) = \sin x \qquad \frac{du}{dx} = \cos x$

$v(x) = \cos x \qquad \frac{dv}{dx} = -\sin x$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$
$$= \frac{\cos x \times \cos x - (\sin x \times -\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} \qquad \text{Since } \sin^2 x + \cos^2 x = 1$$
$$= \sec^2 x$$

Derivatives of trigonometric functions

At the beginning of this chapter we indicated you should be aware of the derivatives of sine and cosine

$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$

Example (7) has shown that we can add to these further derivatives of basic trigonometric functions. The full list follows.

function	derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\cot x \operatorname{cosec} x$
$\sec x$	$\tan x \sec x$
$\cot x$	$-\operatorname{cosec}^2 x$

Example (10)

Write all of the above in the form $\frac{d}{dx}f(x) = f'(x)$

Solution

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\operatorname{cosec} x = -\cot x \operatorname{cosec} x$$
$$\frac{d}{dx}\sec x = \tan x \sec x$$
$$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

Example (11)

Given $f(x) = \sqrt{3}(\tan x + \sec x)$ find $f'(60°)$ giving your answer in surd form.

Solution

$$f(x) = \sqrt{3}(\tan x + \sec x)$$
$$f'(x) = \sqrt{3}(\sec x + \sec x \tan x) = \sqrt{3}\sec x(1 + \tan x) = \frac{\sqrt{3}}{\cos x}(1 + \tan x)$$

When $x = 60°$

$$f'(60°) = \frac{\sqrt{3}}{\cos(60°)}(1 + \tan(60°)) = \frac{2}{\sqrt{3}}(1 + \sqrt{3}) = 2 + \frac{2}{\sqrt{3}}$$

Example (12)

Prove

$(a)\qquad \frac{d}{dx}\text{cosec } x = -\cot x \text{cosec } x$

$(b)\qquad \frac{d}{dx}\sec x = \tan x \sec x$

$(c)\qquad \frac{d}{dx}\cot x = -\text{cosec}^2 x$

Solution

$$(a)\qquad \begin{aligned}\frac{d}{dx}\text{cosec } x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{-\cos x}{\sin^2 x}\\ &= -\frac{\cos x}{\sin x}\times\frac{1}{\sin x}\\ &= -\cot x \text{cosec } x\end{aligned}$$

$$(b)\qquad \begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= \frac{\sin x}{\cos^2 x}\\ &= \tan x \sec x\end{aligned}$$

$$(c)\qquad \begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)\\ &= \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}\\ &= \frac{-\sin^2 x - \cos^2 x}{\cos^2 x}\\ &= -\frac{1}{\cos^2 x}\\ &= -\text{cosec}^2 x\end{aligned}$$

Example (13)

Differentiate $x \tan x^2$

Solution

This is both a product and chain

$$\begin{aligned}\frac{d}{dx}x\tan x^2 &= 1\times\tan x^2 + x\times\frac{d}{dx}\tan x^2 && \text{Product rule}\\ &= 1\times\tan x^2 + x\left(\sec^2\left(x^2\right)\times 2x\right) && \text{Chain rule } \frac{d}{dx}\tan x^2 = \sec^2\left(x^2\right)\times 2x\\ &= \tan x^2 + 2x^2\sec^2\left(x^2\right)\end{aligned}$$

Proofs

We now proceed to prove all the formulae for the product, chain and quotient rules form the definition of the derivative

(1) **Product rule**

To prove $(f \times g)' = f' \times g + f \times g'$

Proof

Let $h(x) = f(x) \times g(x)$

Then

$$h'(x) = \lim_{\delta x \to 0}\left\{\frac{h(x+\delta x) - h(x)}{\delta x}\right\} \qquad \text{where } \delta x \text{ is a small increment in } x$$

$$= \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x)g(x+\delta x) - f(x)g(x)}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x)g(x+\delta x) - f(x+\delta x)g(x) + f(x+\delta x)g(x) - f(x)g(x)}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0}\left\{f(x+\delta x)\left(\frac{g(x+\delta x) - g(x)}{\delta x}\right) + g(x)\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)\right\}$$

$$= \lim_{\delta x \to 0}\left\{f(x+\delta x)\left(\frac{g(x+\delta x) - g(x)}{\delta x}\right)\right\} + \lim_{\delta x \to 0}\left\{g(x)\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)\right\}$$

$$= \lim_{\delta x \to 0}\{f(x+\delta x)\}\lim_{\delta x \to 0}\left\{\left(\frac{g(x+\delta x) - g(x)}{\delta x}\right)\right\} + g(x)\lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$$

$$= f(x)g''(x) + g(x)f'(x)$$

(2) **Chain rule**

To prove $(fg)' = (f'g) \times g'$ or $\dfrac{du}{dx} = \dfrac{du}{dv} \times \dfrac{dv}{dx}$

where $u(x) = u(v(x))$

Proof

We will prove this in the form $\dfrac{du}{dx} = \dfrac{du}{dv} \times \dfrac{dv}{dx}$.

Let $u(x) = u(v) = u(v(x))$

This line is really the key to the proof and the concept. The function u is a function of v, so may be written $u = u(v)$, but v is a function of x, so $v = v(x)$. Then replacing v in

$u = u(v)$ we get $u = u(v(x))$. That also means that u is a function of x, so can write $u = u(x)$ as well. What the formula

$$\frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$$

says is that the rate of change of u with respect to x is equal the rate of change of u with respect to v multiplied by the rate of change of v with respect to x. To prove this, consider the definition of the derivative from first principles, which is

$$\frac{du}{dx} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} \right)$$

where δx is a small increase in x, and δu is a small increase in u. Until we take the limit δx and δu can be treated as small finite quantities, so can be multiplied or divided. Let δv be the small change that is caused in $v(x)$ when x is increased by δx. Since these are all small finite quantities we have

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta x} \times \frac{\delta v}{\delta v} \qquad \text{Equivalent to mulitiplying by 1}$$
$$= \frac{\delta u}{\delta v} \times \frac{\delta v}{\delta x}$$

So

$$\begin{aligned} \frac{du}{dx} &= \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} \right) \\ &= \lim_{\delta x \to 0} \left\{ \frac{\delta u}{\delta v} \times \frac{\delta v}{\delta x} \right\} \\ &= \lim_{\delta x \to 0} \left\{ \frac{\delta u}{\delta v} \right\} \times \lim_{\delta u \to 0} \left\{ \frac{\delta v}{\delta x} \right\} \\ &= \frac{du}{dv} \times \frac{dv}{dx} \end{aligned}$$

(3) **Quotient rule**

To prove $\left(\frac{f}{g} \right)' = \frac{f' \times g - f \times g'}{g^2}$

Proof

We begin by writing $\left(\frac{f}{g} \right)'$ as $\left(\frac{f}{g} \right)' = \left(f \times g^{-1} \right)'$.

However, the symbol g^{-1} is ambiguous. It could be interpreted as either the reciprocal $g^{-1} = \frac{1}{g}$ or as the inverse of g. Usually the symbol g^{-1} means the inverse, but in this context we use it to denote the reciprocal. Then $g^{-1} = \frac{1}{g}$ is a composite function and by the chain rule

$$\left(g^{-1}\right)' = -g^{-2} \times g' = \frac{g'}{g^2}$$

Armed with this result we can substitute into the product rule $\left(f \times h\right)' = f' \times h + f \times h'$ where $h = g^{-1} = \frac{1}{g}$ to get

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f \times g^{-1}\right)' \\ &= f' \times g^{-1} + f \times -g^{-2} \times g' \\ &= \frac{f'}{g} - \frac{f \times g'}{g^2} \\ &= \frac{f' \times g - f \times g'}{g^2}\end{aligned}$$

So the quotient rule is really a special form of the product rule and every problem that can be solved by means of the quotient rule may be solved directly with the product rule.