Direct Integration

Prerequisites

You should already understand that integration is the reverse process of differentiation.

$$g(x) = \int f(x)\,dx \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} f(x) = g'(x)$$

You have also already met the following functions and their derivatives

function	derivative
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

function	derivative
$\tan x$	$\sec^2 x$
$\text{cosec}\, x$	$-\cot x \,\text{cosec}\, x$
$\sec x$	$\tan x \sec x$
$\cot x$	$-\text{cosec}^2 x$

Example (1)

Reverse the table above to fill in the missing entries in the table below

function	integral
x^n	$\frac{x^{n+1}}{n+1} + c \quad n \neq 1$
e^x	
$\frac{1}{x}$	
$\sin x$	
$\cos x$	

function	integral
$\sec^2 x$	
$\cot x \,\text{cosec}\, x$	
$\tan x \sec x$	
$\text{cosec}^2 x$	

Solution

function	integral
x^n	$\frac{1}{n+1}x^{n+1}+c \quad n \neq 1$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln x + c$
$\sin x$	$\cos x + c$
$\cos x$	$-\sin x + c$

function	integral
$\sec^2 x$	$\tan x + c$
$\cot x \operatorname{cosec} x$	$-\operatorname{cosec} x + c$
$\tan x \sec x$	$\sec x + c$
$\operatorname{cosec}^2 x$	$-\cot x + c$

The technique of "direct integration" to find integrals is simply the idea of searching *by trial and error* for a function that reverses the process of differentiation. You should have already learnt to directly integrate simple polynomial functions.

Example (2)

Integrate $x^{\frac{3}{2}} - \frac{6}{x^2}$ with respect to *x*.

Solution

Let $f(x) = x^{\frac{3}{2}} - \frac{6}{x^2}$.

We have to integrate two functions. Let us do this separately and then add the results at the end.

For $f_1(x) = 2x^{\frac{1}{2}}$

Try $\quad x^{\frac{5}{2}} \quad \Rightarrow \quad \frac{d}{dx}x^{\frac{5}{2}} = \frac{5}{2}x^{\frac{3}{2}}$

Try $\quad \frac{2}{5}x^{\frac{3}{2}} \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{2}{5}x^{\frac{3}{2}}\right) = \frac{2}{5} \times \frac{5}{2}x^{\frac{3}{2}} = x^{\frac{3}{2}} \qquad (\checkmark)$

For $f_2(x) = 3x^{-3}$

Try $\quad x^{-1} \quad \Rightarrow \quad \frac{d}{dx}x^{-1} = -x^{-2}$

Try $\quad 6x^{-1} \quad \Rightarrow \quad \frac{d}{dx}6x^{-1} = 6\times -x^{-2} = 6x^{-2} = \frac{6}{x^2} \quad (\checkmark)$

Thus

$$\int x^{\frac{3}{2}} - \frac{6}{x^2}\,dx = \frac{2}{5}x^{\frac{3}{2}} + \frac{6}{x} + c$$

Finally, you should also have learnt how to differentiate products, chains and quotients by the rules

(1) Product rule

$(f\times g)' = f'\times g + f\times g'$ $\qquad$ $\frac{d}{dx}(u\times v) = \frac{du}{dx}\times v + \frac{dv}{dx}\times u$

(2) Chain rule

$(fg)' = (f'g)\times g'$ $\qquad$ $\frac{du}{dx} = \frac{du}{dv}\times\frac{dv}{dx}$ where $u(x) = u(v(x))$

(3) Quotient rule

$\left(\frac{f}{g}\right)' = \frac{f'\times g - f\times g'}{g^2}$ $\qquad$ $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\times\frac{du}{dx} - u\times\frac{dv}{dx}}{v^2}$

Example (3)

Differentiate $e^{-2x}\cos 3x$

Solution

$$\begin{aligned}\frac{d}{dx}e^{-2x}\cos 3x &= \left(\frac{d}{dx}e^{-2x}\right)\cos 3x + e^{-2x}\left(\frac{d}{dx}\cos 3x\right) && \text{Product rule}\\ &= -2e^{-2x}\times\cos 3x + e^{-2x}\times -3\sin 3x && \text{Chain rule}\\ &= -2e^{-2x}\cos 3x - 3e^{-2x}\sin 3x\end{aligned}$$

Extending direct integration

The same method of direct integration can now be extended to cover a wider range of functions. It is simply a question of working one's way through a series of examples.

Example (4)

Find $\int \cos 5x\,dx$

Solution

Try $\sin 5x \Rightarrow \frac{d}{dx}\sin 5x = 5\cos 5x$

Try $\frac{1}{5}\sin 5x \Rightarrow \frac{d}{dx}\frac{1}{5}\sin 5x = \frac{1}{5}\times 5\cos 5x = \cos 5x \quad (\checkmark)$

Hence

$$\int \cos 5x\,dx = \frac{1}{5}\sin 5x + c$$

Henceforth we shall sometimes omit the various trials and show only the result. We can evaluate definite integrals by direct integration.

Example (5)

Evaluate $\int_1^{1.5} e^{3x}\,dx$ giving your answer to 4 significant figures.

Solution

$$\int_1^{1.5} e^{3x}\,dx = \left[\frac{1}{3}e^{3x}\right]_1^{1.5} = \frac{1}{3}e^{4.5} - \frac{1}{3}e^{3} = 23.3105... = 23.31 \text{ (4 s.f.)}$$

The limits in exact integrals involving trigonometric functions are usually given in radians.

Example (6)

Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin(-2x)\,dx$.

Solution

The limits involve fractions of π and this indicates that they are measured in radians not degrees.

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin(-2x)\,dx &= \left[\frac{1}{2}\cos(-2x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \frac{1}{2}\cos\left(-\frac{2\pi}{3}\right) - \frac{1}{2}\cos\left(-\frac{\pi}{2}\right) \\ &= \frac{1}{2}\times -\frac{1}{2} - 0 \\ &= -\frac{1}{4}\end{aligned}$$

Example (7)

If $\frac{dy}{dx} = \sec^2 2x + \operatorname{cosec}^2 3x$ find y

Solution

$$\frac{dy}{dx} = \sec^2 2x + \operatorname{cosec}^2 3x$$

$$\int\left(\sec^2 2x + \operatorname{cosec}^2 3x\right)dx = \int \sec^2 2x dx + \int \operatorname{cosec}^2 3x dx$$

$$= \frac{\tan 2x}{2} - \frac{\cot 3x}{3} + c$$

The form $\frac{a}{(bx+c)^n}$

Certain integrands take the form $\frac{a}{(bx+c)^n}$ where *a*, *b*, *c* are real numbers and *n* is an integer $n \neq 1$.

This is a form of the basic polynomial function $y = x^n$ whose integral is

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c$$

so it can be directly integrated.

Example (8)

Find $\int \frac{9}{(3x-1)^5}dx$

Solution

It may be instructive to show various trials. The function that we are integrating takes the form $\frac{1}{u^5}$ where $u = 3x-1$ so we try first $u^{-4} = \frac{1}{u^4}$.

Try $(3x-1)^{-4}$ $\Rightarrow$ $\frac{d}{dx}(3x-1)^{-4} = -4(3x-1)^{-5}$

Try $-\frac{9}{4}(3x-1)^{-4}$ $\Rightarrow$ $\frac{d}{dx} -\frac{9}{4}(3x-1)^{-4} = -\frac{9}{4} \times -4(3x-1)^{-5} = \frac{9}{(3x-1)^5}$ (✓)

Hence

$$\int \frac{9}{(3x-1)^5}dx = -\frac{9}{4(3x-1)^4} + c$$

The form $\dfrac{f'(x)}{f(x)}$

Certain integrands take the form $\dfrac{f'(x)}{f(x)}$ for a given function *f*.

Example (9)

(*a*) Using the chain rule differentiate each of the following

(*i*) $y = \ln(x+3)$

(*ii*) $y = \ln(x^2)$

(*iii*) $y = \ln(f(x))$ $\quad f(x) > 0$

(*b*) What is the integral of $\dfrac{f'(x)}{f(x)}$?

Solution

(*a*) (*i*) $y = \ln(x+3)$ $\quad x > -3$

$$\frac{dy}{dx} = \frac{1}{x+3} = \frac{1}{y}$$

(*ii*) $y = \ln(x^2)$

$$\frac{dy}{dx} = \frac{2x}{x^2} = \frac{y'}{y}$$

(*iii*) $y = \ln(f(x))$ $\quad f(x) > 0$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

(*b*) Since integration is the reverse operation to differentiation, the last result **suggests** that

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$$

From example (9) we **conjecture** that $\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$. However, we must be careful here, because logarithm is a function that is **not defined** for negative values. Therefore, if $f(x)$ is negative then $\ln(f(x))$ is not defined. Suppose $f(x) > 0$ then, as in example (9)

$$\frac{d}{dx}\ln|f(x)| = \frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

But if $f(x) < 0$ then $|f(x)| = -f(x)$ and

$$\frac{d}{dx}\ln|f(x)| = \frac{d}{dx}\ln(-f(x)) = \frac{-f'(x)}{-f(x)} = \frac{f'(x)}{f(x)}$$

So whether $f(x) > 0$ or $f(x) < 0$ we have $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$, but this shows that the correct result for the reverse process is

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c\ .$$

The introduction of the modulus sign prevents the possibility arising of introducing an undefined and hence meaningless function as a result of attempting to integrate a function. Questions may test your understanding of this result

Example (10)

Find $\int \frac{3}{(6x+2)}dx$

Solution

Try $\ln|6x+2|$ $\Rightarrow$ $\frac{d}{dx}\ln(6x+2) = \frac{6}{(6x+4)}$

Try $\frac{1}{2}\ln|6x+2|$ $\Rightarrow$ $\frac{d}{dx}\frac{1}{2}\ln(6x+2) = \frac{1}{2}\times\frac{6}{(6x+4)} = \frac{3}{(6x+4)}$ (✓)

Hence

$$\int \frac{3}{(6x+2)}dx = \frac{1}{2}\ln|6x+2| + c$$

Example (11)

Find $\int \left(\frac{5x}{(1-x^2)} + \frac{4}{(1-x)^3}\right)dx$

Solution

$$\int \left(\frac{5x}{(1-x^2)} + \frac{4}{(1-x)^3}\right)dx = \int \frac{5x}{(1-x^2)}\ dx + \int \frac{4}{(1-x)^3}dx$$

$$= -\frac{5}{2}\ln|1-x^2| - \frac{1}{2(1-x)^2} + c$$


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