Eigenvalues and Eigenvectors

Introduction

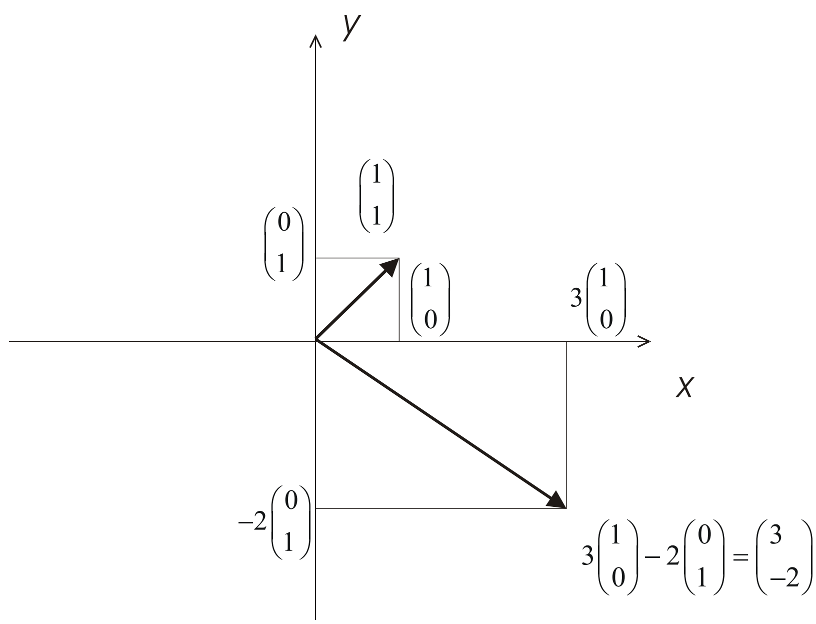
A matrix represents a transformation of the plane.

Some matrices represent rotations of the plane; some matrices represent reflections of the plane, and some matrices represent shears.

Another general class of matrices represent stretches of the *x,y*-plane. A simple example of this is the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

This matrix stretches the *x*-axis 3 times and stretches the *y*-axis twice, but in the opposite direction.



What happens to the *x* and *y*-axes also determines what happens to every other vector lying in the plane. For instance, since

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The matrix A has the effect of scaling this vector by 3 times in the x-direction and by –2 times in the y-direction.

$$A\begin{pmatrix}1\\1\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix} + A\begin{pmatrix}0\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$$

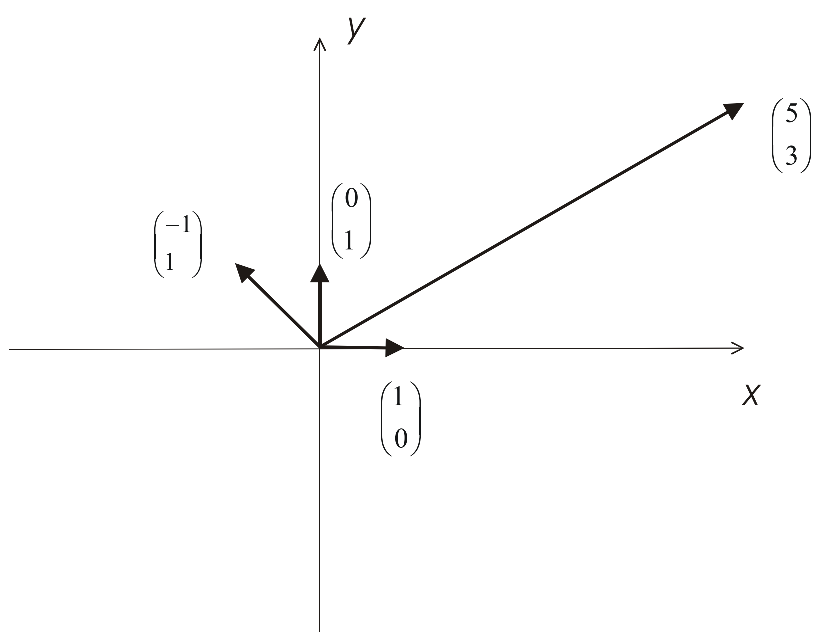
In other words, if we know that a matrix has the effect of stretching vectors along the two coordinate axes, we are then able to describe the effect of the matrix on any vector lying in the plane.

However, pretty obviously, the case of stretches along *both* or even either the x and y-axes is the exception, rather than the rule. We seek a more general theory that will apply to all matrices that represent some form of systematic stretching of the plane.

To illustrate this, consider the matrix

$$A = \begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}$$

If we try out at random some vectors and see what this matrix does to them, we may be at a loss to explain its effect on the plane as a whole. For example, this matrix sends the unit vector in the x-direction, $(1,0)$ to the point $(5,3)$, and the unit vector in the y-direction, $(0,1)$ to the vector $(-1,1)$, but this information does not really help us to visualise the effect of the transformation on the plane as a whole.



After some trial and error, however, the student may discover that there are lines, like the x and y-axes described in the first example, along which the effect of the transformation is simply to stretch (and possibly reflect, since a reflection is a

negative stretch). For example, one such line which is mapped to itself by the matrix A, lies in the direction $(1,3)$. Operating by on this vector by means of the matrix A

$$A\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}5&-1\\3&1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}=2\begin{pmatrix}1\\3\end{pmatrix}$$

by obtain another vector which is simply a multiple of the first. So, in the direction represented by the vector

$$\underline{\mathbf{x}}_1=\begin{pmatrix}1\\3\end{pmatrix}$$

the matrix A has the effect of stretching the plane. Another vector which has this property is

$$\underline{\mathbf{x}}_2=\begin{pmatrix}1\\1\end{pmatrix}$$

Along this vector we obtain another scalar multiple of itself by operating with A

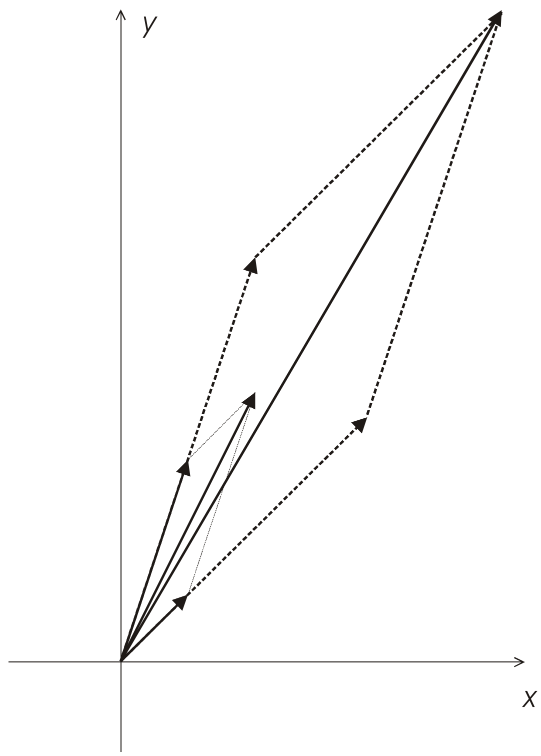
$$\begin{pmatrix}5&-1\\3&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}$$

Since any two vectors are sufficient to define the whole 2-dimensional plane – they form a basis for the plane (or more technically, they form a basis for $\mathbb{R}^2$), the entire effect of the transformation represented by the matrix A, can be summarised by saying that A stretches a vector by a factor of 2 in the direction of the vector

$$\underline{\mathbf{x}}_1=\begin{pmatrix}1\\3\end{pmatrix}$$

and by a factor of 4 in the direction of the vector

$$\underline{\mathbf{x}}_2=\begin{pmatrix}1\\1\end{pmatrix}$$



Since we have uncovered by trial and error this simple geometric description of the effect of a whole class of matrices on the *x,y*-plane, we seek a systematic algebraic technique to discover those lines that are mapped to themselves by such matrices, and the scale factors by which they are mapped.

Determining eigenvalues and eigenvectors by use of the characteristic equation

Firstly, we introduce the relevant new terminology. An invariant point is one which is its own image under a transformation represented by a matrix, *A*. An invariant line is one for which all points on the line map to points also on the line. The points on the line are not necessarily invariant.

An eigenvector, $\underline{\mathbf{x}}$, is a vector that is parallel to an invariant line, so that the invariant line can be described by the equation

$$\underline{\mathbf{r}} = t\underline{\mathbf{x}}$$

Since the line is invariant, but the vector, $\underline{\mathbf{x}}$, is possibly scaled by the matrix *A*, then

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$$

where λ is a scale factor. In fact, this scale factor may be a complex number, but at this level we will examine only matrices that have real valued scale factors. The scale factor is called an eigenvalue.

The largest number of eigenvalues and eigenvectors that a matrix can have is equal to its dimension.

Thus, a 2×2 matrix can have at most 2 eigenvalues and 2 eigenvectors, and a 3×3 matrix can have at most 3 eigenvalues and 3 eigenvectors.

It turns out that a matrix can have two distinct eigenvectors but only 1 eigenvalue – that is, the scale factor is repeated or the same for the two eigenvectors.

Since not every square matrix describes a scaling, even in the broadest sense given here, it does not follow that we can find eigenvalues and eigenvectors for every square matrix.

The reason why eigenvalues and eigenvectors are useful, is because matrices describe linear transformations. So what happens to any basis vectors completely describes the whole transformation. This can be expressed by

If

$$A\underline{\mathbf{x}} = \lambda_1 \underline{\mathbf{x}}$$
$$A\underline{\mathbf{y}} = \lambda_2 \underline{\mathbf{y}}$$

then

$$A\left(\alpha \underline{\mathbf{x}} + \beta \underline{\mathbf{y}}\right) = A\left(\alpha \underline{\mathbf{x}}\right) + A\left(\beta \underline{\mathbf{y}}\right) = \alpha A \underline{\mathbf{x}} + \beta A \underline{\mathbf{y}} = \alpha \lambda_1 \underline{\mathbf{x}} + \beta \lambda_2 \underline{\mathbf{y}}$$

When eigenvectors exist the following procedure will find them.

Firstly, we form what is called the characteristic equation for the matrix.

$$\left(A - \lambda I\right)\underline{\mathbf{x}} = 0$$

is called the characteristic equation for the matrix *A*.

For example, for our matrix

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

The characteristic equation is given by

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = 0$$
$$(5-\lambda)(1-\lambda) + 3 = 0$$

Next, we solve this equation to find the values of λ that satisfy it. For a 2×2 matrix we can expect at most 2 such values, which we will designate λ_1 and λ_2. These will be the eigenvalues of the matrix.

To illustrate this process on our matrix, we have already seen

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = 0$$
$$(5-\lambda)(1-\lambda) + 3 = 0$$

Now solving this

$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda - 2)(\lambda - 4) = 0$$
$$\lambda_1 = 2 \text{ or } \lambda_2 = 4$$

In the next stage we observe that since these are eigenvalues for the matrix, then they must satisfy the equation

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$$

so we substitute the values of λ_1 and λ_2 to obtain at most two eigenvectors. If the eigenvalues are distinct then it can be shown that there will be two distinct eigenvectors. Continuing with our example

$$\text{when } \lambda = 4$$
$$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$
$$5x - y = 4x$$
$$3x + y = 4y$$

Note that at this stage we have obtained two equations in x and y, but we are expecting only one relationship between them. Both equations must describe the same relationship, and if they do not, then an error has been made earlier on in the calculation. In this case, solving either equation gives the relationship

$$y = x$$

which defines the eigenvector

$$\underline{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

repeating the process for the second eigenvalue, when

$$\lambda = 2$$

$$\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$5x - y = 2x$$

$$3x + y = 2y$$

$$y = 3x$$

$$\underline{\mathbf{x}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

So the matrix, A, is the matrix that stretches the x,y plane by a factor of 4 in the direction of

$$\underline{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and by a factor of 2 in the direction of

$$\underline{\mathbf{x}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The set of eigenvectors of A are also called the spectrum of A.

Diagonalisation

The matrix that scales by a factor of 4 along the x-axis, and by a factor of 2 along the y-axis is

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

Since there are zero entries for D along the oblique diagonal, its eigenvalues can be immediately read. Matrix D has the same eigenvalues as matrix A of our earlier example, where

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

In fact, there is a sense in which A and D are the same matrix, since, if the basis vectors of the plane that D transforms were the same as the basis vectors as the plane that A transforms, they would perform the same transformation. That is to same, A can be regarded as having been obtained from D by a change of basis. For A the x-axis of D has become the eigenvector

$$\underline{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the y-axis of D has become the eigenvector

$$\underline{\mathbf{x}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

where these eigenvectors are still described in the coordinate system of D.

Because of this intimate relationship between the matrix A and the matrix D which has only non-zero entries along its main diagonal, we define them to be *similar*.

An $n \times n$ matrix A is called similar to an $n \times n$ matrix D if

$$A = X^{-1}DX$$

for some nonsingular matrix D.

This transformation is called the similarity transformation.

The matrix X that enables the matrix A to be represented by its similar diagonal matrix D is effectively a change of basis. Since it maps the basis vectors of D to the eigenvectors of A, it is simply the matrix

$$X = \left(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2\right)$$

This gives a square matrix, because the two eigenvectors are 2×1 column matrices. For example, the basis transformation for A above is

$$X = \left(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2\right) = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

The inverse matrix is

$$X^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

Hence the matrix A can be represented by

$$A = XDX^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

Power matrices

An immediate benefit of the technique of diagonalising a given matrix, can be seen in the ability to use diagonal matrices to find higher powers of that matrix.

Firstly, let us suppose that we have a diagonal representation of a matrix,

$$A = XDX^{-1}$$

Then

$$\begin{aligned} A^2 &= \left(X^{-1}DX\right)^2 \\ &= \left(X^{-1}DX\right)\left(X^{-1}DX\right) \\ &= X^{-1}DXX^{-1}DX \\ &= X^{-1}DDX \\ &= X^{-1}D^2X \end{aligned}$$

Generalising this result

$$A^n = X^{-1}D^nX$$

The proof would be by mathematical induction.

Example

(i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$$

and hence find a diagonal matrix such that

$$D = X^{-1}AX$$

(ii) Find A^3

Solution

(i) $$A = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(-1-\lambda) - 8 = 0$$

$$3 + 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 + 4\lambda - 5 = 0$$

$$(\lambda + 5)(\lambda - 1) = 0$$

$\lambda_1 = -5$ or $\lambda_2 = 1$

When $\lambda_1 = -5$

$$\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$-3x + 2y = -5x$$

$$4x - y = -5y$$

$$y = -x$$

which gives the eigenvector

$$\underline{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

when $\lambda_2 = 1$

$$\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$-3x + 2y = x$

$4x - y = y$

$2y = 4x$

$y = 2x$

which gives the eigenvector

$\underline{\mathbf{x}}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$X^{-1} = \dfrac{1}{\det X}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \dfrac{1}{2+1}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

D is the diagonal matrix, and we can write it down directly from having obtained the eigenvalues, which were $\lambda = -5$ and $\lambda = 1$, hence

$D = \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}$

However, we can confirm that $D = X^{-1}AX$ by

$D = X^{-1}AX = \dfrac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$D = \dfrac{1}{3}\begin{pmatrix} -10 & 5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} -15 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) If $A = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$ find A^3

$A = \dfrac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$A^3 = \dfrac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}^3\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \dfrac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -125 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \dfrac{1}{3}\begin{pmatrix} -125 & 1 \\ 125 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \dfrac{1}{3}\begin{pmatrix} -149 & 126 \\ 152 & -123 \end{pmatrix}$