Elementary Trigonometric Identities

Prerequisites

The topic of elementary trigonometric identities builds specifically on knowledge of the solution of equations involving trigonometric expressions and on knowledge of the use of the quadratic formula for the solution of quadratic equations.

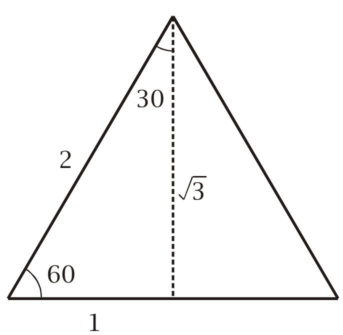
Example (1)

Solve the equation

$\sin x = -\frac{\sqrt{3}}{2}$ for $0 \le x \le 360°$

Solution

Whilst this problem could be solved with the assistance of a calculator, that is in fact unnecessary, and you should recognise $\frac{\sqrt{3}}{2}$ as a ratio derived from one of the special triangles found frequently in problems involving trigonometry. This is the triangle



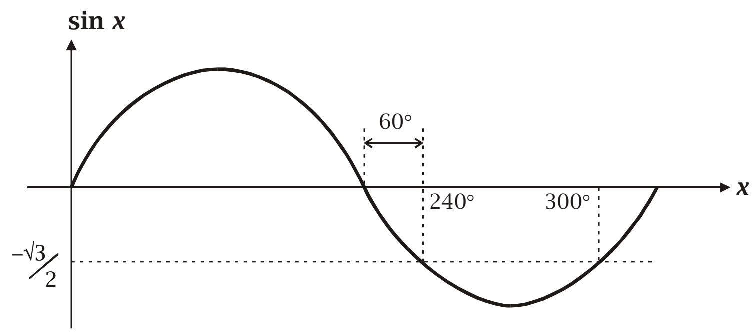
which demonstrates the relationships

$\cos 30 = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60 = \frac{1}{2}$

$\sin 30 = \frac{1}{2}$ $\quad$ $\sin 60 = \frac{\sqrt{3}}{2}$

$\tan 30 = \frac{1}{\sqrt{3}}$ $\quad$ $\tan 60 = \sqrt{3}$

So $\sin x = \frac{\sqrt{3}}{2}$ has solution $x = 60°$ in the interval $0° \le x \le 90°$. We are in fact asked to solve $\sin x = -\frac{\sqrt{3}}{2}$ for $0 \le x \le 360°$. To solve this we begin by sketching a graph of the sine function for $0 \le x \le 360°$, and draw the line $y = -\frac{\sqrt{3}}{2}$ to intersect with it.



$x = 240°$ or $x = 300°$ for $0 \le x \le 360°$

Example (2)

(*a*) Solve $3x^2 - 2x - 2 = 0$.

(*b*) Hence solve

$3\sin^2\theta - 2\sin\theta - 2 = 0$ for $0 \le \theta \le 360°$

Solution

(*a*) On substitution into the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we have

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 3 \times -2}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

Hence

$$x = \frac{1+\sqrt{7}}{3} = 1.2152\ldots \text{ or } x = \frac{1-\sqrt{7}}{3} = -0.5485$$

(*b*) $3\sin^2\theta - 2\sin\theta - 2 = 0$ is obtained from $3x^2 - 2x - 2 = 0$ by means of the substitution $x = \sin\theta$. Hence

$\sin\theta = 1.2152\ldots$ or $\sin\theta = -0.5485\ldots$

However θ cannot take a value outside the range $-1 \le \theta \le +1$ so $\sin\theta = 1.2152\ldots$ has no solution. The equation $\theta = \sin^{-1}(-0.5485\ldots)$ has solutions

$\theta = 213.3°$ or $\theta = 326.7°$ $(0.1°)$ for $0 \le \theta \le 360°$.

Elementary trigonometric identities

The two elementary trigonometric identities are

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

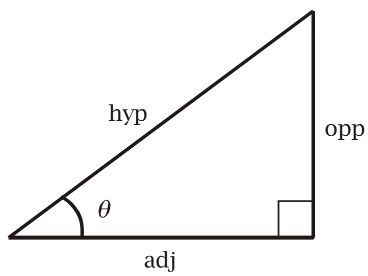
(2) $\sin^2\theta + \cos^2\theta \equiv 1$

Proofs

(1) The relationship between tan, sin and cos.

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

This relationship holds because of the definitions of tan, sin and cos.

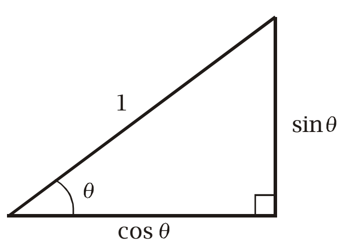


$$\tan\theta \equiv \frac{opp}{adj} \equiv \frac{opp/hyp}{adj/hyp} \equiv \frac{\sin\theta}{\cos\theta}$$

(2) Relationship between sin and cos.

$$\sin^2\theta + \cos^2\theta \equiv 1$$

This relationship follows from Pythagoras's theorem.



Allowing the length of the hypotenuse to be 1, the lengths of the adjacent and opposite sides are $\sin\theta$ and $\cos\theta$ respectively, which follows from substitution into

$\sin\theta \equiv \dfrac{opp}{hyp}$ and $\cos\theta \equiv \dfrac{adj}{hyp}$ respectively.

The trigonometric ratios are *ratios* and do not depend on the size of the triangle, so these results would hold whatever the length of the hypotenuse. On substitution into Pythagoras's theorem

$\sin^2\theta + \cos^2\theta \equiv 1$

Identities and equations

The trigonometric identities are relationships between functions. That means should strictly we should use the sign $\equiv$ to express this relationship. Observe the $\equiv$ rather than the equals sign = in these expressions.

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

(2) $\sin^2\theta + \cos^2\theta \equiv 1$

The sign $\equiv$ denotes equivalence between functions, and represents the idea that both sides of the identity are true for all values of the functions. The symbol = represents an equality between numbers. Thus $\equiv$ stands for a relationship between functions, and = stands for a relationship between numbers. Let us clarify this further. In the expression

$\sin^2\theta + \cos^2\theta \equiv 1$

the equivalence $\equiv$ says that the left-hand side, $\sin^2\theta + \cos^2\theta$, is *always* equal to the right-hand side, 1, whatever the value of the angle, θ. In the expression

$\sin\theta = \dfrac{1}{2} \qquad 0 \le \theta \le 90°$

the equals = says that the left-hand side, $\sin\theta$ is equal to the right-hand side, $\dfrac{1}{2}$, for (in this case) *one* value of the angle θ in the range $0 \le \theta \le 90°$. The equation

$\sin\theta = \dfrac{1}{2} \qquad 0 \le \theta \le 90°$

can be *solved* to find that particular value. Thus

$\sin\theta = \dfrac{1}{2}$

$\theta = \sin^{-1}\left(\dfrac{1}{2}\right) = 30° \qquad \text{for } 0 \le \theta \le 90°$

It is *not* always true, for all values of θ that $\sin\theta = \dfrac{1}{2}$. For example, $\sin 45° = \dfrac{1}{\sqrt{2}}$. This means that the statement $\sin\theta \equiv \dfrac{1}{2}$ (note the equivalence $\equiv$ in this) is *false*.

Example (3)

Show by counter-example that the statement $\sin\theta \equiv \frac{1}{2}$ is false.

Solution

Let $\theta = 45°$, then $\sin 45° = \frac{1}{\sqrt{2}} \neq \frac{1}{2}$.

Remark

We could have substituted any value other than $\theta = 30°$ in the interval $0 \le \theta \le 90°$ to provide a counter-example. The statement $\sin\theta = \frac{1}{2}$ is only *true* for $\theta = 30°$ in the interval $0 \le \theta \le 90°$.

Example (4)

(*a*) Solve

$$\cos^2\theta = \cos\theta \qquad 0 \le \theta \le 90°$$

(*b*) Show by counter-example that the statement

$$\cos^2\theta \equiv \cos\theta$$

is false.

Solution

(*a*)

$$\cos^2\theta = \cos\theta$$
$$\cos^2\theta - \cos\theta = 0$$
$$\cos\theta(\cos\theta - 1) = 0$$
$$\cos\theta = 0 \text{ or } \cos\theta = 1$$
$$\theta = 90° \text{ or } \theta = 0° \text{ for } 0 \le \theta \le 90°$$

Remark

The statement $\cos^2\theta = \cos\theta$ is only true if θ is such that $\cos^2\theta = \cos\theta$ becomes equivalent to $0 = 0$ or $1 = 1$.

(*b*) We may substitute any value for θ other than 0° or 90° to obtain a counter example. Denote the left-hand side of $\cos^2\theta \equiv \cos\theta$ by LHS and the right-hand side by RHS.

$$\text{LHS} = \cos^2\theta \qquad \text{RHS} = \cos\theta$$

Let $\theta = 60°$ then

$$\text{RHS} = \cos\theta = \frac{1}{2}$$

$$\text{LHS} = \cos^2\theta = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \neq \text{RHS}$$

Using the elementary trigonometric identities

We have proven the elementary trigonometric identities

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (2) $\sin^2\theta + \cos^2\theta \equiv 1$.

Let us now illustrate their use.

Example (5)

Solve $\dfrac{\sin\theta}{\cos\theta} = 1$ for $0 \le \theta \le 90°$

Solution

Replace $\dfrac{\sin\theta}{\cos\theta}$ by $\tan\theta$ using the first identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ to obtain

$$\tan\theta = 1$$
$$\theta = \tan^{-1} 1$$
$$\theta = 45° \quad \text{for } 0 \le \theta \le 90°$$

The two identities can be rearranged to give several other identities. For example

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

can be rearranged to give

$$\sin\theta = \tan\theta\cos\theta .$$

Example (6)

The two elementary trigonometric identities are

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

(2) $\sin^2\theta + \cos^2\theta \equiv 1$

Find all the possible rearrangements of these two equations by means of algebraic operations without addition of new terms.

Solution

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

$$\cos\theta \equiv \frac{\sin\theta}{\tan\theta}$$

$$\sin\theta \equiv \tan\theta\cos\theta$$

(2) $\sin^2\theta + \cos^2\theta \equiv 1$

$$\sin^2\theta \equiv 1 - \cos^2\theta$$

$$\cos^2\theta \equiv 1 - \sin^2\theta$$

$$\sin\theta \equiv \sqrt{1 - \cos^2\theta}$$

$$\cos\theta \equiv \sqrt{1 - \sin^2\theta}$$

Advice

These rearrangements are particularly useful in all areas of mathematics. However, do not try to learn them as such, but *commit to memory* both the identities

(1) $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

(2) $\sin^2\theta + \cos^2\theta \equiv 1$

and be able to *rearrange them in the given context* to solve a particular problem.

Example (7)

Solve $6\sin^2\theta = \cos\theta + 5$ for $0 \le \theta \le 360°$

Solution

$$6\sin^2\theta = \cos\theta + 5$$

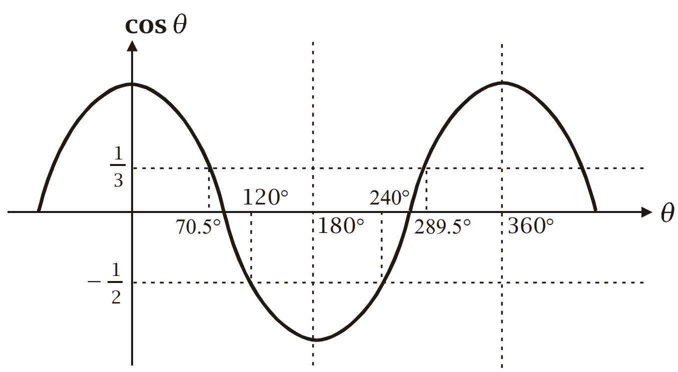
$$6\left(1 - \cos^2\theta\right) = \cos\theta + 5$$

$$6 - 6\cos^2\theta - \cos\theta - 5 = 0$$

$$6\cos^2\theta + \cos\theta - 1 = 0$$

$$(3\cos\theta - 1)(2\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{3} \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$



$\theta = 70.5°, 120°, 240°$ or $289.5°$ $(0.1°)$ for $0 \le \theta \le 360°$

Further trigonometric ratios

You are familiar with the three trigonometric ratios

$$\sin\theta \equiv \frac{opp}{hyp} \qquad \cos\theta \equiv \frac{adj}{hyp} \qquad \tan\theta \equiv \frac{opp}{adj}.$$

We now add to this list three other ratios that are the reciprocals of the above. These are cosecant, secant and cotangent, abbreviated to cosec, sec and tan respectively and defined by

$$\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta} \qquad \sec\theta \equiv \frac{1}{\cos\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}.$$

Calculators generally do not carry extra buttons for these functions for the reason that they would be superfluous. Problems involving cosec, sec and cot are converted into problems involving sin, cos and tan respectively.

Example (8)

Solve

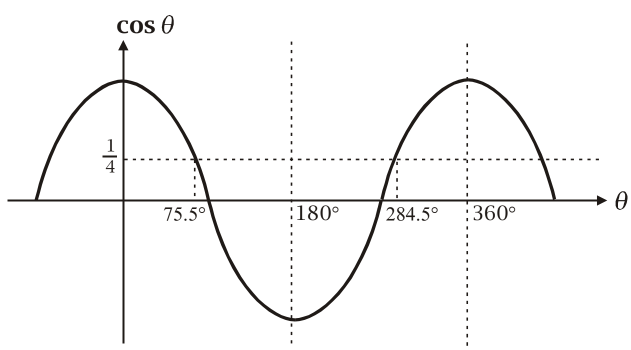
$\sec\theta = 4 \qquad$ for $0 \le \theta \le 360°$

Solution

$\sec\theta = 4$

$$\frac{1}{\cos\theta} = 4$$

$$\cos\theta = \frac{1}{4}$$



$\theta = 75.5°$ or $284.5°$ $(0.1°)$ for $0 \le \theta \le 360°$

Deriving further trigonometric identities

From the relationship $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ we can derive the relationship

(1) $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$

Proof

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\frac{1}{\tan\theta} \equiv \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$$

From the relationship $\sin^2\theta + \cos^2\theta = 1$ two further trigonometric identities involving cosec, sec and tan can be demonstrated.

(2) $\tan^2\theta + 1 \equiv \sec^2\theta$

Proof

$$\sin^2\theta + \cos^2\theta \equiv 1$$

On dividing both sides by $\cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

(3) $1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$

Proof

$$\sin^2\theta + \cos^2\theta \equiv 1$$

On dividing both sides by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin\theta} \equiv \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$$

The last two equations may naturally be rearranged to give other forms of these identities.

(2)′ $\tan^2\theta + 1 \equiv \sec^2\theta$

$$\tan^2\theta \equiv \sec^2\theta - 1$$

$$\sec^2\theta - \tan^2\theta \equiv 1$$

(3)′ $1+\cot^2\theta \equiv \text{cosec}^2\theta$

$\cot^2\theta \equiv \text{cosec}^2\theta - 1$

$\text{cosec}^2\theta - \cot^2\theta \equiv 1$

Advice

You can always recover the equation $\tan^2\theta + 1 \equiv \sec^2\theta$ from the equations $\sin^2\theta + \cos^2\theta \equiv 1$ and $\sec\theta \equiv \frac{1}{\cos\theta}$. So *commit to memory* $\sin^2\theta + \cos^2\theta \equiv 1$ and the definitions of sec, cosec and cot and recover any of the other identities in the context of a question by means of algebraic rearrangements. It is more economical to learn the proofs rather than try to remember all the results. Even the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ can be remembered from the basic definitions of sin, cos and tan as $\sin\theta \equiv \frac{opp}{hyp}$, $\cos\theta \equiv \frac{adj}{hyp}$ and $\tan\theta \equiv \frac{opp}{adj}$ and by recalling the proof.

Example (9)

Solve

$2\text{cosec}\theta + 5 = 3\cot^2\theta$ for $0 \le \theta \le 360°$

Solution

$2\text{cosec}\theta + 5 = 3\cot^2\theta$

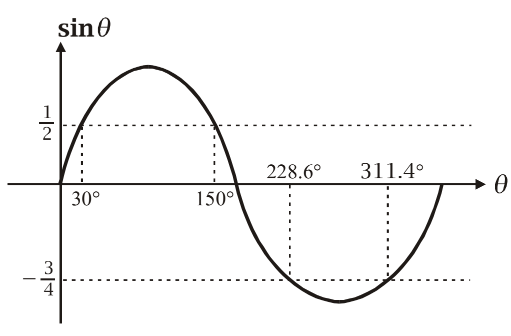
$2\text{cosec}\theta + 5 = 3\left(\text{cosec}^2\theta - 1\right)$

$3\text{cosec}^2\theta - 2\text{cosec}\theta - 8 = 0$

$\left(3\text{cosec}\theta + 4\right)\left(\text{cosec}\theta - 2\right) = 0$

$\text{cosec}\theta = -\frac{4}{3}$ or $\text{cosec}\theta = 2$

$\sin\theta = -\frac{3}{4}$ or $\sin\theta = \frac{1}{2}$ since $\text{cosec}\theta \equiv \frac{1}{\sin\theta}$



$\theta = 30°,\ 150°,\ 228.6°$ or $311.4°\ \left(0.1°\right)$ for $0 \le \theta \le 360°$