Energy Conversions

Prerequisites

You should be familiar already with

(1) Hooke's law

When an elastic material is subject to an applied force, F, the extension, x, produced obeys the law $F = kx$ where k is the *spring constant,* which is also given by

$$k = \frac{\lambda}{l_0}$$

where λ = Young's modulus, and l_0 is the natural length of the spring.

(2) The concept of work and its definition as

$$\text{work} = \text{force} \times \text{distance}\ ,$$

together with the principle of conservation of energy and applications of that principle to deal with energy conversions between gravitational potential energy, kinetic energy and work done against friction.

The purpose of this chapter is to combine your knowledge of both fields.

Principle of Conservation of Energy

When an object is dropped from a height it gathers speed. This is because the object's gravitational potential energy is converted to kinetic energy. Let us ignore air-resistance. Then the rule that lost gravitational potential energy is converted to kinetic energy is an example of the application of a general principle of conservation of energy. None of the energy is lost – all of the gravitational potential energy is converted to kinetic energy.

Principle of Conservation of Energy

A system is said to be *closed* when no external forces act on it. The total energy of a closed system remains constant.

When a force acts on a system it either introduces or withdraws energy from it. So from a closed system no energy can leak, and total energy within a closed system remains constant.

Elastic Potential Energy

A stretched elastic string, or a spring under extension or compression, has energy. For example, a stretched rubber band has the potential to pull two objects together. This energy is called *elastic potential energy*. It is given by

$$E = \frac{1}{2}kx^2$$

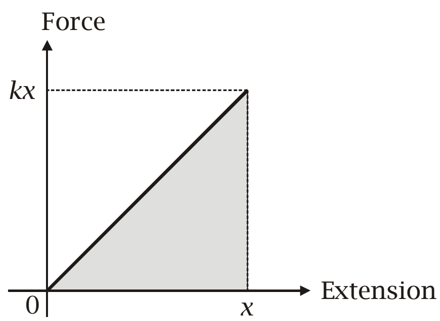
where x is the extension or compression and k is the spring constant. As before, $k = \frac{\lambda}{l_0}$ where

λ = Young's modulus
l_0 = natural length

This formula is in fact derived from the definition of work energy as

work = force $\times$ distance work = $F \times d$

Suppose a spring is extended by a distance x. The applied force producing this extension is $F = kx$.



The work done is the area under the force function = Area of triangle in the diagram.

$$W = \frac{1}{2}kx \times x = \frac{1}{2}kx^2$$

We can also show this using integration. Work is the integral of the applied force function up from 0 to the maximum extension x. Let the force function be $F = ks$, then

$$W = \int_0^x ks\ ds$$

$$W = W = \int_0^x ks\ ds = \left[\frac{k}{2}s^2\right]_0^x = \frac{1}{2}kx^2$$

Note the distinction between x, which is the actual extension, and s which the variable representing all the extensions of which the spring is capable, including x.

Conversions between gravitational, kinetic and elastic energy

We can apply the principle of conservation of energy to closed systems where there are conversions between the following three forms of energy

Kinetic energy $E = \frac{1}{2}mv^2$

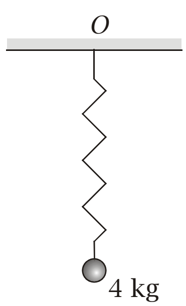
Gravitational potential energy $U = mgh$

Elastic potential energy $E = \frac{1}{2}kx^2$

Our first example examines the case where gravitational potential energy is entirely converted to elastic potential energy.

Example (1)

The diagram shows a particle of mass 4 kg attached by means of a light elastic spring to a fixed point O.



The natural length of the spring is 0.6 m. The particle is hanging in equilibrium 1.3 m vertically below O.

(*a*) Calculate the modulus of elasticity of the spring.

(*b*) Determine the elastic potential energy stored in the spring.

Solution

(*a*) The extension of the spring is $x = l - l_0 = 1.3 - 0.6 = 0.7$ m

The particle is in equilibrium, so the tension in the spring is equal to its weight.

$T = W$

$kx = mg \Rightarrow \quad k = \frac{mg}{x} = \frac{4 \times 9.8}{0.7} = 56$

$k = \frac{\lambda}{l_0} \quad \Rightarrow \quad \lambda = k \times l_0 = 56 \times 0.6 = 33.6$ N

(*b*) $E = \frac{1}{2}kx^2 = \frac{1}{2} \times 56 \times (0.7)^2 = 13.72 = 13.7$ J (3 s.f.)

Example (2)

A particle of mass 2 kg is attached to one end of a light elastic string of natural length 2 m and modulus of elasticity λ Newtons. The other end of the string is attached to a fixed point, *A*. The particle is released from rest at *A*, and, moving under gravity, comes to instantaneous rest at a distance 3 m vertically below *A*. Find the value of λ.

Solution

In this question the particle starts at the fixed point *A*, so the loss of gravitational potential energy is the distance between *A* and the point where it comes to a point of instantaneous rest, which is 3 m below *A*. At this point the particle has no kinetic energy.

Gain of elastic potential energy = Loss of gravitational potential energy

$$\frac{1}{2}kx^2 = mgh$$

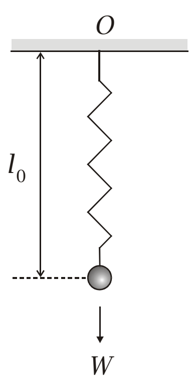
$$m = 2,\ x = l - l_0 = 3 - 2 = 1,\ h = 3$$

$$\frac{1}{2}k \times (1)^2 = 2 \times 9.8 \times 3$$

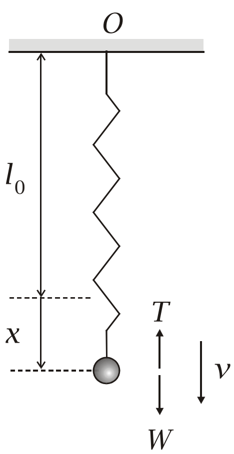
$$k = 117.6$$

$$k = \frac{\lambda}{l_0} \quad \Rightarrow \qquad \lambda = k \times l_0 = 117.6 \times 2 = 235.2 = 235 \text{ N } (3 \text{ s.f.})$$

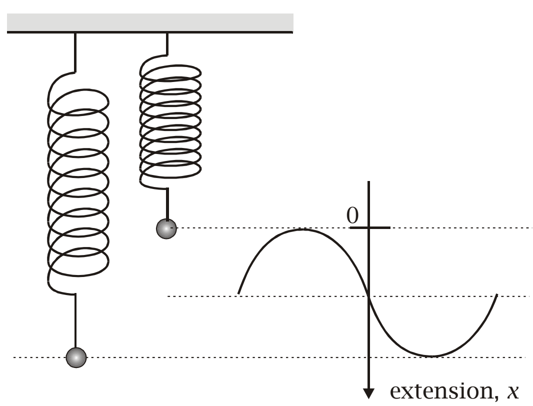
So far we have examined only the case where the gravitational potential energy of a suspended sphere is entirely converted to elastic potential energy. The following diagram shows a sphere of mass *m* kg suspended by means of a light elastic spring to the ceiling running through a smooth hook.



Initially the sphere is held at rest at the natural length of the spring. It is then released. At that instantaneous moment the sphere is subject only to the force of gravity, and its weight, $W = mg$, pulls it down. As it falls it loses gravitational potential energy. Part of this energy is converted into elastic potential energy in the cord, and part is converted into the kinetic energy of the falling sphere. As it falls the tension in the spring increases. A point is reached where the tension in the spring is equal in magnitude (and opposite in direction) to the weight of the sphere, so the resultant force is zero.



However, it is still moving downwards because it has a velocity v metres per second determined by the kinetic energy it has gained as a result of falling x metres. Beyond this point the tension in the spring continues to increase, so the tension is greater than the weight of the sphere and the sphere starts to slow down. It is still losing gravitational potential energy and gaining elastic potential energy. Its kinetic energy is decreasing. Eventually, it reaches a point where all the kinetic energy previously gained has been converted to elastic potential energy. At this point the tension is still greater than the weight and the tension starts to pull the sphere back up again.



We can see that the system would in fact oscillate, and if no energy were lost from the system the oscillations would carry on forever. In practice at every stage some of the energy of the system is lost as friction, the spring (and atmosphere) heat up and eventually the whole system comes to rest at the equilibrium point where the tension in the spring and weight of the sphere are equal. The study of the oscillations of this system is left for a subsequent chapter. For the present we are concerned solely with the energy conversions taking place. As the object oscillates, the gravitational potential energy, elastic potential energy and kinetic energy will be inter-converted. Assuming that the system is not losing energy in the form of friction, the total energy remains the same by the principle of conservation of energy. Hence

total energy	=	gravitational potential energy	+	kinetic energy	+	elastic potential energy
E	=	mgh	+	$\frac{1}{2}mv^2$	+	$\frac{1}{2}kx^2$

Example (3)

A light elastic string has modulus 24 N and natural length 1.2 m. One end is attached to a fixed point *O*. The other is attached to a particle of 0.6 kg. The particle initially held at *O* is released from rest.

(*a*) Find the elastic potential energy when the extension of the string is *x* meters.

(*b*) Find the maximum extension of the string.

(*c*) Calculate the speed of the particle when it is 1.3 m below *O*.

Solution

(*a*) elastic potential energy $= \frac{1}{2}kx^2$

$$k = \frac{\lambda}{l_o} = \frac{24}{1.2} = 20$$

elastic potential energy $= \frac{1}{2} \times 20 \times x^2 = 10x^2$

(*b*) At maximum extension kinetic energy = 0

All the gravitational potential energy is converted to elastic potential energy.

loss of gravitational potential energy = gain of elastic potential energy

$$mgh = \frac{1}{2}kx^2$$

$$k = 20$$

$$0.6 \times 9.8 \times (l_0 + x) = 0.5 \times 20 \times x^2$$

$$0.6 \times 9.8 \times (1.2 + x) = 10x^2$$

$$10x^2 - 5.88x - 7.056$$

$$x = \frac{5.88 \pm \sqrt{5.88^2 + 4 \times 10 \times 7.056}}{20}$$

$$= \frac{5.88 \pm 17.799...}{20}$$

$$= 1.1839..$$

$$= 1.2 \text{ m} \quad (2 \text{ s.f.})$$

The negative solution to this equation would refer to the other point of equilibrium where kinetic energy is also zero, so is not relevant here.

(*c*) In this part of the question the extension is given by

$x = 0.1$

loss of gravitational potential = gain of kinetic energy + gain of elastic energy

$$mg(l_0 + x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$0.6 \times 9.8 \times (1.2 + 0.1) = \frac{1}{2} \times 0.6v^2 + 10 \times 0.1^2$$

$$v^2 = \frac{0.6 \times 9.8 \times 1.3 - 10 \times (0.1)^2}{0.3}$$

$$v^2 = 25.146...$$

$$v = \sqrt{25.146...} = 5.014...$$

$$v = 5.0 \text{ ms}^{-1} \quad (2 \text{ s.f.})$$