# Equilibrium of non-parallel, non-concurrent coplanar forces

## Prerequisites

You should be familiar with (1) moments and problem solving involving parallel non-concurrent coplanar forces that are in equilibrium; (2) contact forces, including the normal reaction at a surface, friction and the coefficient of friction.

**Current and non-concurrent forces**

When two or more forces act on an object lie in the same plane, they are said to be *coplanar*. When the forces pass through the same point they are said to be *concurrent*. In the case of forces applied at a distance from a pivot then the forces are *non-concurrent*. Non-concurrent forces produce turning effects.

**Example (1)**

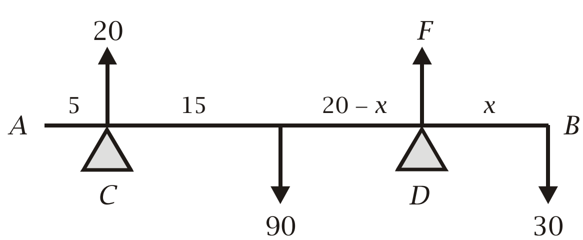
The diagram shows a uniform rod $AB$ of length 40 cm and weight 90 N resting horizontally on two smooth supports $C$ and $D$ where $AC$ = 5 cm and $BD$ = $x$ cm. A load of 30 N is placed at $B$.


(*a*) Given that the reaction at $C$ is 20 N, find the reaction at $D$ and the value of $x$.

(*b*) Find the greatest value of $x$ for which the rod remains in equilibrium.

Solution

(*a*) Let the reaction at $D$ be $F$.



The value of $F$ must be sufficient to support the weight of the rod and the load less the reaction at $C$, which is 20 N. So

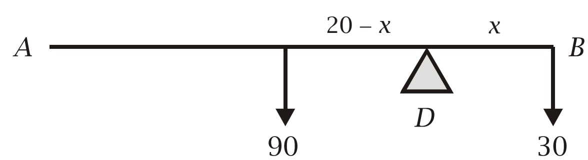
$$20 + F = 90 + 30$$
$$F = 100 \text{ N}$$

To find $x$ we can take moments at either $C$ or $D$. Taking moments at $C$.

Sum of clockwise moments = Sum of anticlockwise moments

$$(90 \times 15) + (30 \times 35) = 100 \times (35 - x)$$
$$\frac{2400}{100} = 35 - x$$
$$x = 11 \text{ cm}$$

(*b*) As $x$ increases the reaction at $C$ decreases, until it is zero. At this point the rod is just balancing on the support at $D$, so in effect the support at $C$ ceases to make a contribution. We can redraw the diagram as follows.



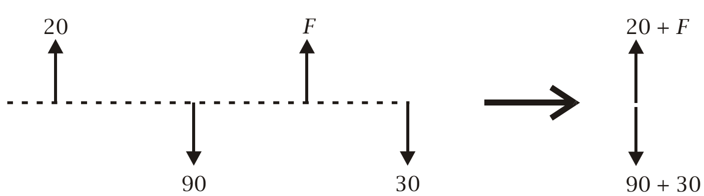

Taking moments at $D$.

Sum of clockwise moments = Sum of anticlockwise moments

$$30x = 90(20 - x)$$
$$120x = 1800$$
$$x = 15 \text{ cm}$$

This example illustrates a number of important principles that are involved in problems of this type.

(1) In part (*a*) we used the fact that all the forces are in static equilibrium to deduce the value of an unknown force.




This part of the problem did not involve moments at all. The fact that the forces are non-concurrent is not required to solve this part of the problem.

(2) When non-concurrent forces are involved we use the principle
Sum of clockwise moments = Sum of anticlockwise moments
to find an unknown quantity.

(3) In the last part of the question information about a *limiting condition* meant that one force disappeared and that we could redraw the diagram and so solve the problem.

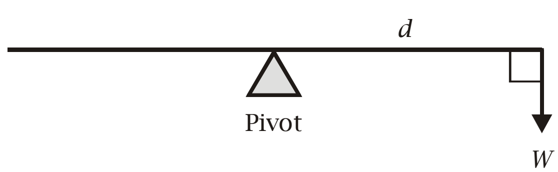
Up to now all the non-concurrent forces that you have met have also been parallel to each other. In the above example all the forces act perpendicularly to the rod.

## Non-parallel non-concurrent forces

You are aware that when forces act at a distance from a pivot they produce a turning effect. We use the terms *moment* or *torque* (they are interchangeable) for this turning effect. The moment is defined to be

Moment = Force $\times$ perpendicular distance of the force from the axis of rotation
$C = F \times d$

In all the problems we have considered so far the force has acted perpendicularly to axis of rotation, so we have not had to consider the relevance of the phrase *perpendicular distance* in this definition above. To illustrate the idea of perpendicular distance, consider a boy sitting on the edge of a seesaw at a distance $d$ metres from the pivot placed at the centre of the see-saw. We will model the boy as a particle of weight $W$ newtons.

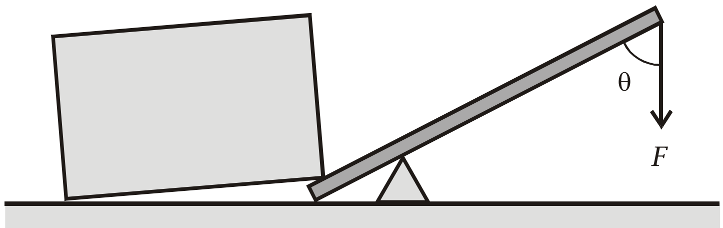



As the diagram shows his weight $(W)$ acts downwards at a right angle to the seesaw and the moment is given by

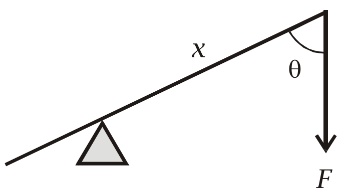
Moment = weight $\times$ the perpendicular distance from the pivot.
$= Fd$

However, when forces do not act at right angles to the axis of rotation, then we also need the concept of a perpendicular distance.

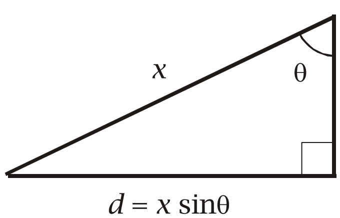
For example, a moment is produced when a metal bar is used to lift a block of stone.




In the above diagram the force $F$ is being applied at an angle to the shaft of the metal bar. We can draw this diagram schematically as follows.




Here $x$ denotes the length of the bar, which we are assuming to be a light rod, that is a rigid rod of no weight. To find the moment produced by the force $F$ it would be incorrect to take the product $Fx$ as this would too big. The correct distance is the perpendicular distance of the force to the pivot.

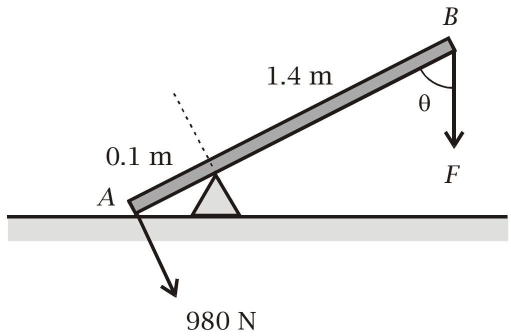



The perpendicular distance is $d = x\sin\theta$ and the moment is

$C = F \times x\sin\theta$

**Example (2)**

The diagram shows a light rod $AB$ of length 1.5 m A force of $F$ N applied at $B$ at an angle $\theta°$ where $\sin\theta = 0.7$ .




There is a pivot 0.1 m under the rod from *A*. Find the minimum force *F* required to lift a load of 980 N acting perpendicularly to the rod at *A*.

Solution

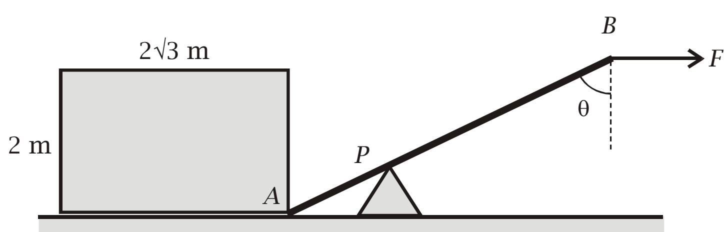
moment at $A$ = moment at $B$

$980 \times 0.1 = F \times 1.4 \sin\theta$

$F = \dfrac{980 \times 0.1}{1.4 \times 0.7} = 100 \text{ N}$

**Example (3)**

The diagram shows a uniform stone slab of mass 1000 kg and dimensions $2\sqrt{3} \times 2$ m.



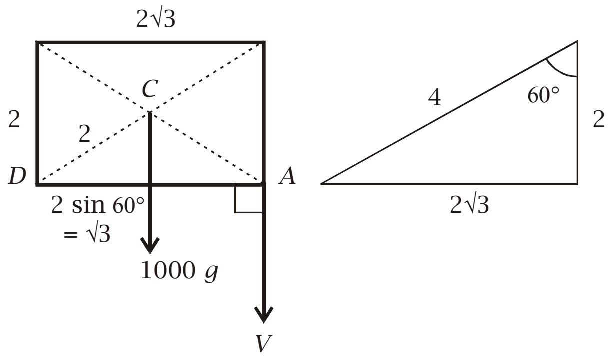

A light rod *AB* has been placed under one corner *A* of slab, the edge of which may be assumed to be lying horizontal as shown in the diagram. The distance $AP = 0.1$ m and the distance $PB = 1.4$ m. A horizontal force *F* is applied at *B*. *AB* makes an angle $\theta°$ with the vertical where $\tan\theta = 0.8$. The slab is on the point of moving.

(*a*) Modelling the slab as a lamina with centre of mass at the point of intersection of the diagonals, find the force exerted by the weight of the slab acting perpendicularly to the horizontal plane at *A*.

(*b*) Find the minimum force *F* required to lift the block.

Solution

(*a*) The centre of mass of the slab lies at the intersection of the diagonals.




The dimensions of the slab make the angle between a diagonal and a vertical 60°. When the slab is lifted it will pivot about the point $D$ shown in the diagram above. The perpendicular distance of the weight of the slab $(W = 1000g)$ is $2\sin 60° = \sqrt{3}$. The moment produced by the slab about $D$ is

$C = 1000g \times 2\sin 60° = \sqrt{3} \times 1000 \times 9.8$ Nm

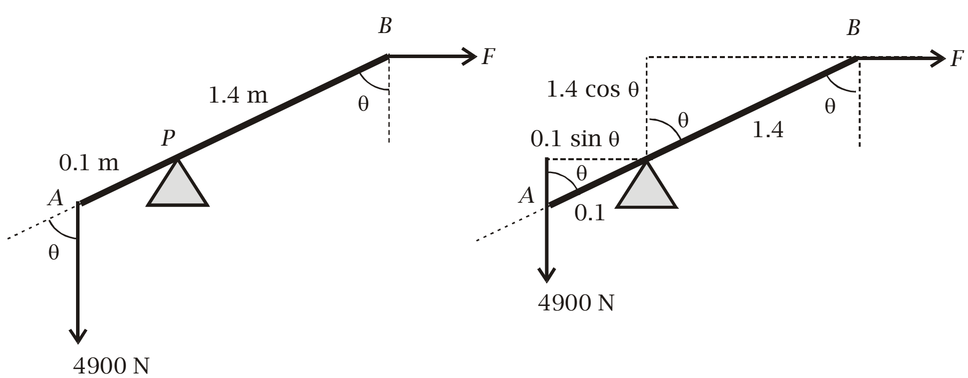
The equivalent force $V$ acting at $A$ is given by

$V \times 2\sqrt{3} = \sqrt{3} \times 1000 \times 9.8$

$V = 4900$ N

(*b*) The perpendicular distance of the 4900 N force from the pivot $P$ is $d_1 = 0.1\sin\theta$.

The perpendicular distance of the force $F$ from the pivot is $d_2 = 1.4\cos\theta$.




The magnitudes of the moments produced by the two forces are equal.

$Vd_1 = Fd_2$

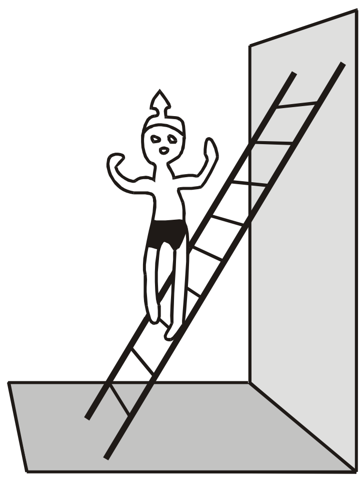
$4900 \times 0.1\sin\theta = F \times 1.4\cos\theta$

$F = \dfrac{4900 \times 0.1}{1.4}\tan\theta = 350 \times 0.8 = 280$ N

# Force diagrams

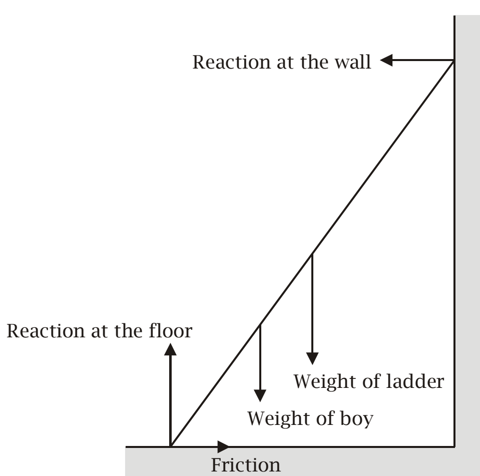
We need to be able to visualise the effect of forces on objects. We do this by means of a *force diagram*.

**Example (4)**



A young boy climbed one-third up a ladder. The wall itself was totally smooth but fortunately for the boy, whose ladder might otherwise have slipped, the floor was rough. By representing the boy as a particle draw two force diagrams showing (*a*) the forces on the ladder and (*b*) the forces acting on the boy.

Solution

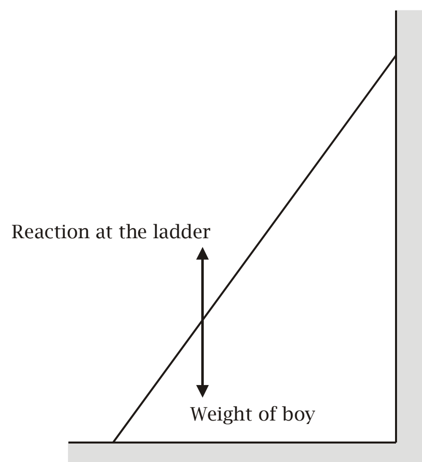



(*a*) The weight of the ladder acts at the centre of the mass of the ladder, which here is the mid-point of the ladder. (We are assuming the ladder may be modelled as a rod of uniform mass.) The boy, standing one third up the ladder causes a downward thrust on the ladder, equal to his weight. The ladder presses against the wall, and this squeezes the wall slightly, so the wall responds by thrusting against the ladder – this is the reaction at the wall. The wall itself is smooth, so there is no friction acting along the wall, and there is only a reaction at the wall, which is perpendicular to the wall. At the foot of the ladder there are two forces acting on the ladder both due to the *contact* made between the ladder and the floor. Firstly, the ladder is pushing into the floor, so the floor responds by pushing back. This is the normal reaction at the floor. Also the ladder is not slipping. This is because the rough end of the ladder is catching the rough surface of the floor, so this rough surface is pushing back. This is the friction, which is shown as acting along the surface of the floor. This is also a contact force. We call this whole arrangement of forces a "system". The entire system is not moving – it is in equilibrium. Therefore, by resolving forces vertically and horizontally

$(\uparrow)$ Weight of ladder + Weight of Toto = Normal reaction at the floor

$(\rightarrow)$ Reaction at the wall = Friction along the surface of the floor

(*b*) If we look at the system from the point-of-view of the boy, there are just two forces acting on him.




Firstly, there is boy's weight, acting downwards; secondly, since the boy is standing on the ladder and pressing down on it, the ladder is responding with a reaction. This reaction is equal and opposite to the boy's weight, since the boy is not moving, and he is in equilibrium.

**Remarks**

(1) Force diagrams depend on the point-of-view. The force diagram from the point-of-view of the ladder is *not* the same as the force diagram from the point-of-view of the boy. The force diagram for the system as a whole, that is, if we included the wall, would have no forces marked on it at all. This is because the system as a whole is not moving (we will ignore for the moment the motion of the Earth). All the forces inside this system cancel out.

(2) In order to draw these force diagrams we make two *modelling assumptions*.

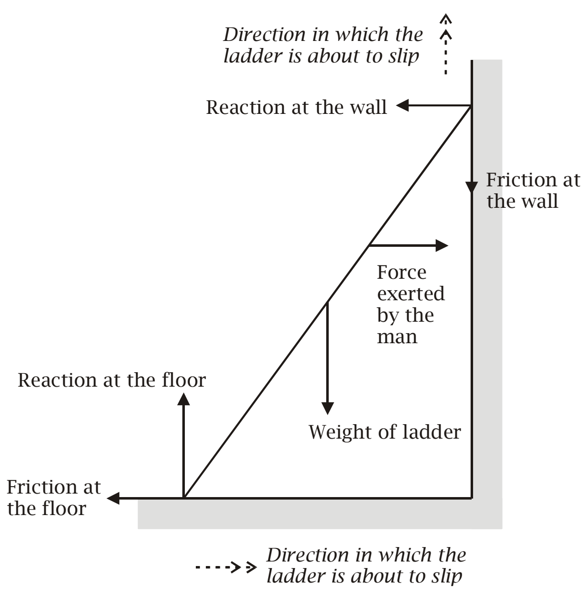
(*a*) That the ladder may be modelled as a rod of uniform mass with centre of mass at the mid-point of the rod.

(*b*) That the boy may be modelled as a particle.

(3) The number and direction of the forces depend upon the conditions given in the question. If the wall is not smooth, then a frictional contact force must be drawn where the ladder meets the wall. The direction of the frictional forces depend on which way the ladder would slip if it were to slip.

**Example (5)**

A ladder *AB* is resting against a rough wall with point *A* in contact with the floor, which is also rough. A man leans against the ladder at a point 2/3rds from *A*, and the ladder is just about to slip. The direction of the force exerted by the man against the ladder is horizontal with the floor and directed towards the wall. Draw a diagram representing the forces acting on the ladder.




This diagram differs from the first diagram in example (4) as follows.

(1) Because the wall is rough a frictional force has been drawn there.

(2) The direction of the friction at the base of the ladder has been reversed. This is because the ladder is about to slip. Now normally, if a ladder resting against a wall slips, it does so by its foot sliding outwards away from the wall. But here the man is exerting a force against the wall, so if it slips the head of the ladder will slide up the wall and the foot will slide along the floor in the direction of the wall. Friction opposes motion so the frictional forces are drawn acting in the opposite direction to the possible motion of the ladder. The obvious moral of this example being that one must be careful with force diagrams, because they are tools for solving problems, you must represent the problem correctly by the number and direction of the forces.

## Problem solving

Using force diagrams, the principle of static equilibrium and the principle of equilibrium of non-concurrent forces (the principle of moments), we are now in a position to solve further problems in statics.

**Example (6)**

A uniform ladder, of mass 40 kg and length 10 m, rests with its top end against a rough vertical wall and its bottom end on rough horizontal ground. The ladder is inclined to the ground at an angle of $\theta$ where $\tan\theta = \frac{4}{3}$. The coefficient of friction between the ladder and the ground is $\mu_1 = 0.4$ and the coefficient of friction between the ladder and the wall is $\mu_2 = 0.2$. A person of mass 60 kg climbs the ladder. Find how far the person can ascend before the ladder slips.

Solution

Let the weight of the ladder be $W = 40g = 40 \times 9.8 = 392$ N

Let the weight of the man be $M = 60g = 60 \times 9.8 = 588$ N

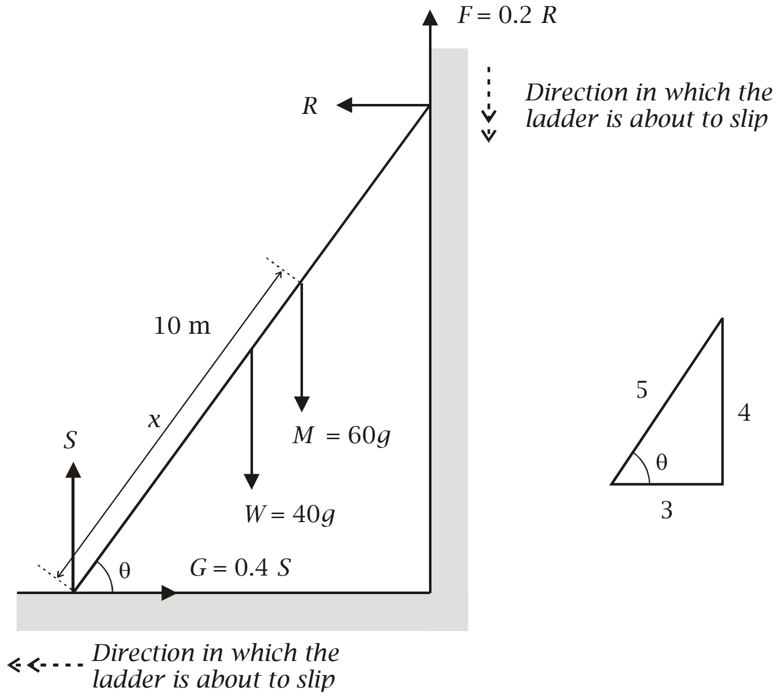
Let the reaction at the wall be $R$

Let the friction at the wall be $F = \mu_2 R = 0.2R$

Let the reaction at the floor be $S$

Let the friction at the floor be $G = \mu_1 S = 0.4S$

Let the position of the man up the ladder measured from the foot at the point where the ladder is about to slip be $x$ metres.




Resolving horizontally and vertically according the principle of static equilibrium

$(\rightarrow)\qquad G = R$

$$0.4S = R$$

$(\uparrow)\qquad S + F = W + M$

$$S + 0.2R = 980$$

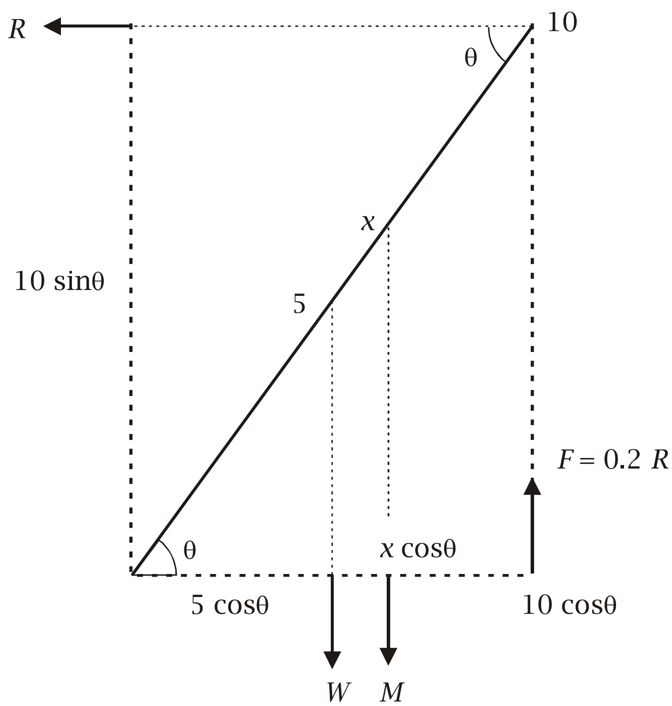
$$S + 0.2(0.4S) = 980$$

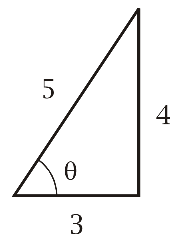
$$1.08S = 980$$

$$S = \frac{980}{1.08}$$

$$R = 0.4 \times \frac{980}{1.08} = \frac{392}{1.08}$$

To solve the problem we must take moments somewhere. Taking moments at the foot of the ladder has the advantage of cancelling out the turning effects of the forces marked *S* and *G* here.




(↻) $W \times 5\cos\theta + M \times x\cos\theta = R \times 10\sin\theta + F \times 10\cos\theta$

Substituting $\cos\theta = \frac{3}{5},\ \sin\theta = \frac{4}{5},\ W = 392,\ M = 588,\ R = \frac{392}{1.08},\ F = 0.2R$

$$\left(392 \times 5 \times \frac{3}{5}\right) + \left(588x \times \frac{3}{5}\right) = \left(\frac{392}{1.08} \times 10 \times \frac{4}{5}\right) + \left(0.2 \times \frac{392}{1.08} \times 10 \times \frac{3}{5}\right)$$

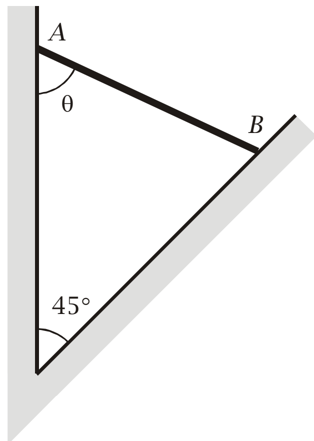
$$5880 + 1764x = \frac{1}{1.08}(15680 + 2352)$$

$$1764x = \frac{18032}{1.08} - 5880$$

$$x = 6.1316... = 6.13 \text{ m} \quad (3 \text{ s.f.})$$

The key to these questions is always to draw the force diagram correctly in the first place, and then, after resolving horizontally and vertically, to choose wisely a point about which to take moments. A suitable point is one where some of the forces have zero moments, thus making the calculations easier. However, if the system is in equilibrium then the sum of the moments about any point whatsoever must be zero.

**Example (7)**



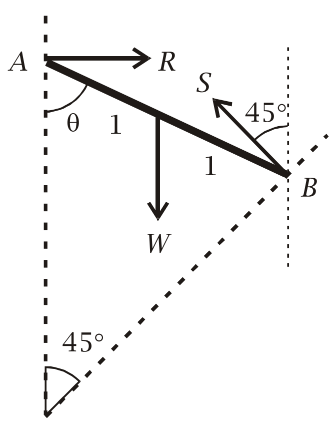

The diagram shows a uniform rod AB of length 2 m resting between a vertical plane and a plane inclined at 45° to the vertical. Calculate the angle $\theta$ if

(*a*) both planes are frictionless.

(*b*) the inclined plane is frictionless but the vertical plane is rough, A is about to slip down and m the coefficient of friction is $\mu = \frac{3}{4}$.

Solution

(*a*) We begin by drawing a force diagram. Since both planes are smooth the contact forces where the rod meets both planes are perpendicular to the planes (normal reactions) and there is no friction.




The important thing to realise about this diagram is that the normal reaction marked *S* acts at 45° to the vertical.

Resolving horizontally and vertically

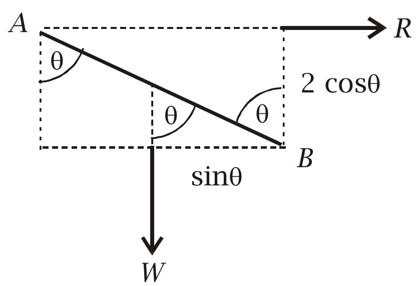
$(\rightarrow)$ $\quad R = S\sin 45°$

$(\uparrow)$ $\quad W = S\cos 45°$

Hence

$$\tan 45° = \frac{R}{W} \quad \Rightarrow \quad R = W$$

Taking moments at $B$



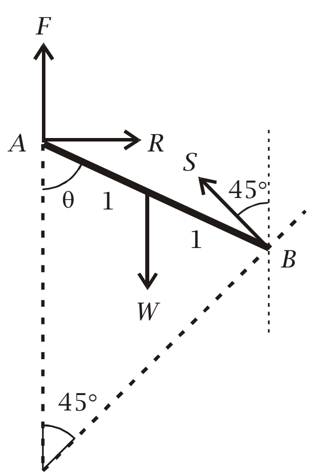

$(\circlearrowright)$ $\quad W\sin\theta = 2R\cos\theta$

$$R\sin\theta = 2R\cos\theta$$

$$\tan\theta = 2$$

$$\theta = \tan^{-1}(2) = 63.4° \ (0.1°)$$

(*b*) The force diagram must now include friction at *A*.




Resolving horizontally and vertically

$(\rightarrow)$ $R = S\sin 45°$

$(\uparrow)$ $W = F + S\cos 45°$

We are given $F = \mu R = \frac{3}{4}R$. Hence

$(\rightarrow)$ $R = S\sin 45°$

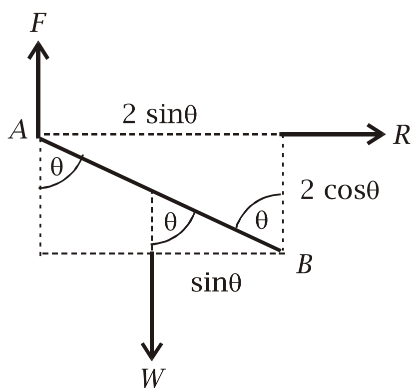
$(\uparrow)$ $W = F + S\cos 45°$

$$S = \frac{R}{\sin 45°}$$

$$W = \frac{3}{4}R + \left(\frac{R}{\sin 45°}\right)\cos 45°$$

$$W = \frac{7}{4}R$$

Taking moments at *B*




$(\circlearrowleft)$ $F \times 2\sin\theta + R \times 2\cos\theta = W\sin\theta$

$$\frac{3}{4}R \times 2\sin\theta + 2R\cos\theta = \frac{7}{4}R\sin\theta$$

$$2\cos\theta = \frac{1}{4}\sin\theta$$

$$\tan\theta = 8$$

$$\theta = \tan^{-1}(8) = 82.9° \ (0.1°)$$