Equilibrium of Rigid Bodies in Contact

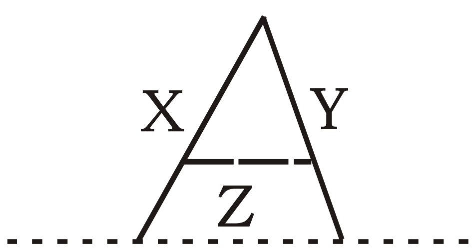
This subject is a further extension of that branch of mechanics that is called statics and is concerned with the equilibrium of static objects – objects that are not moving.

The theory that is assumed, apart from a general understanding of algebra, is:

1. The process of resolving forces into components – often horizontal and vertical components – and the principal that the sum of forces that are in static equilibrium is zero.
2. For a body in static equilibrium the sum of all moments about any point or axis is zero.
3. Basic knowledge about centres of mass. There is no new theory involved. This topic is concerned with an extension of the given theory of statics to more complicated examples.
4. Newton's Third Law: every action has an equal and opposite reaction.
5. Knowledge of the coefficient of friction and understanding of toppling and sliding.

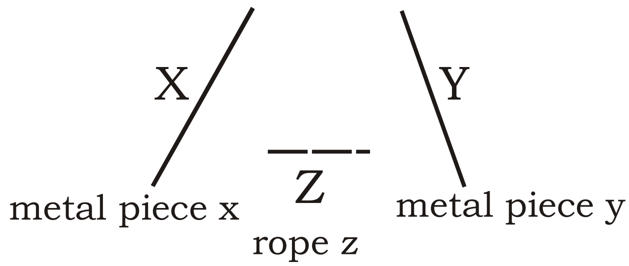
What is new is that success in problem-solving in this area requires (1) the development of a certain way of viewing objects – the ability to see objects in part to whole relationships – and (2) the development of physical intuition and understanding of forces. Success in this topic requires a clear grasp of existing theory and visual and physical insight.

To illustrate these ideas we consider an asymmetrical step-ladder.



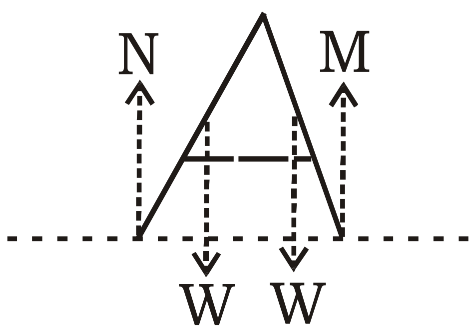
The ladder comprises two metal pieces X and Y, hinged together and standing on a smooth floor, held together by a rope, Z. Y is shorter than X. Since the floor is smooth there are no frictional forces at the floor, consequently there are only the normal reactions to consider at the floor. In this illustration we shall also suppose that both vertical pieces, X and Y, have the same weight W.

The spatial insight required is illustrated as follows. The ladder can be viewed either as a WHOLE or as three separate PARTS. The whole is shown above, but as parts, the object is a composite of:

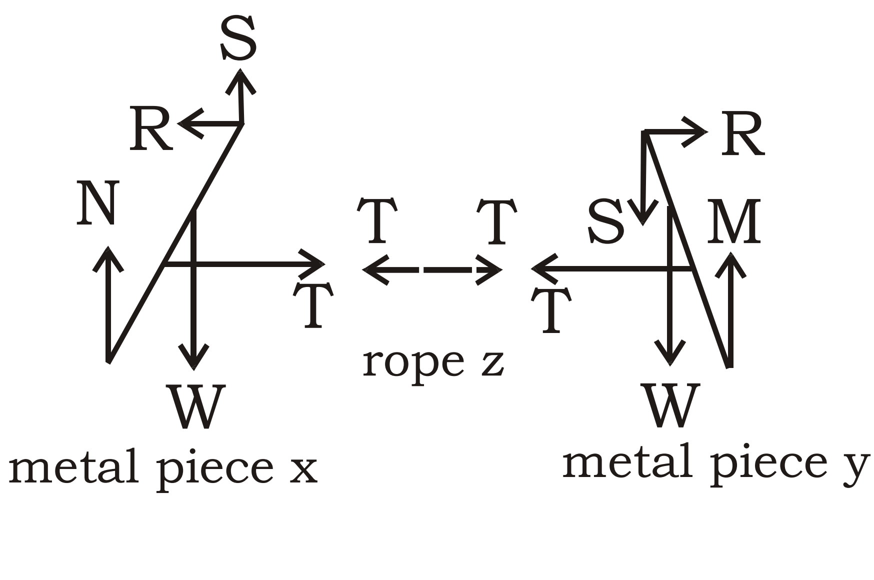


When we look at the object as a whole we consider only the external forces acting on it as a whole. These are just the weight of each and the reactions at the floor. Because the rope is "light" we shall assume it has no weight. Since the floor is "smooth" there are no frictional forces on the floor.

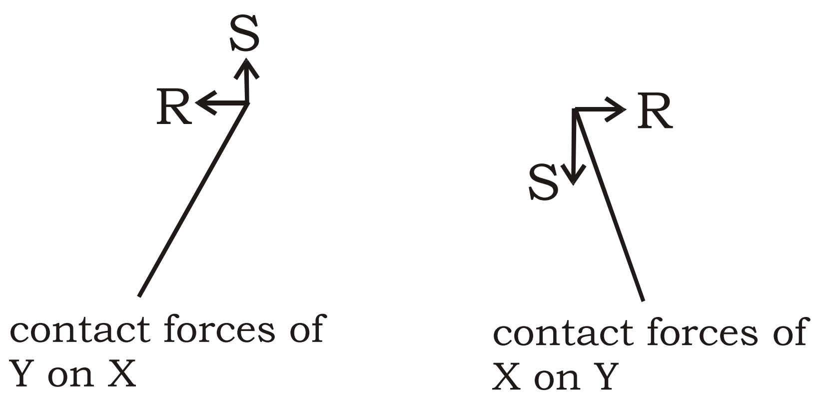
Thus, the forces acting on the ladder as a whole are:



We ignore all the internal forces such as the tension in the rope – these all cancel out when the object is viewed as a whole. However, a force that is internal to an object viewed as a whole may be external to PART of the object. The forces acting on each point of the ladder are:



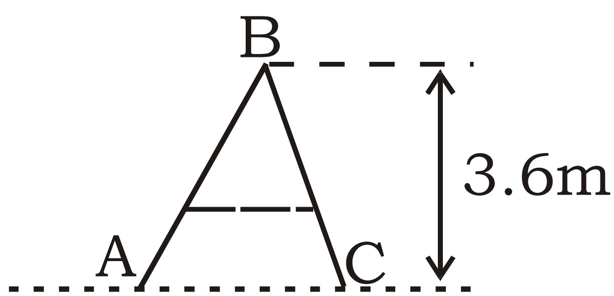
The metal piece X is being acted upon by a tension in the rope pulling in the direction of Y. At the hinge the piece Y is pushing against X. This contact force has been resolved into two parts – a horizontal component and a vertical component. Physical intuition is required to see which way the forces are acting. Piece Y is "pushing" piece X upwards. To see why observe that since Y is shorter than X it will be bearing none of the common weight, hence, it must be pulling X upwards. The contact forces acting on X due to Y must be equal in size to the contact forces acting on Y due to X. This is because the object as a whole is not moving, and is an application of Newton's Third Law. That law also tells us about the direction of the forces – the contact forces between two objects must be equal in size and opposite in direction.



The contact forces of Y on X must be equal in size and opposite in direction to the contact forces of X on Y.

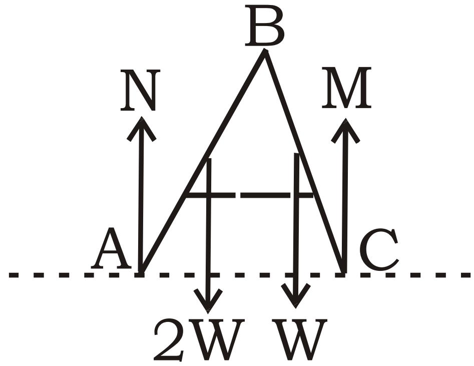
We will now develop this into a full example.

Example (1)



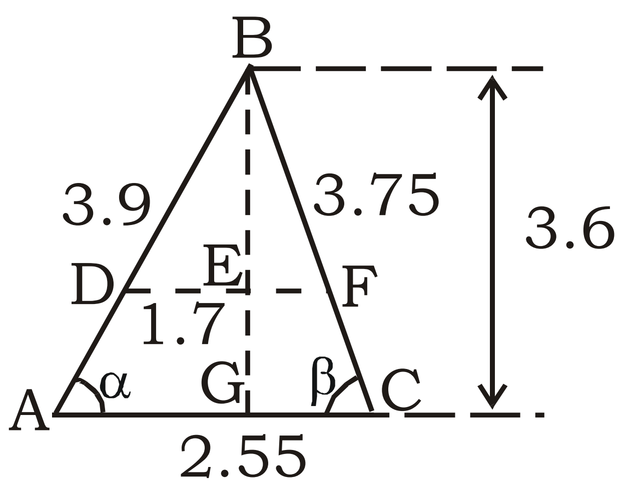
A step-ladder is made of two metal pieces, AB and BC. The length of AB is 3.9m and that of BC is 3.75m, when open the height of the ladder is 3.6m above the ground. The metal pieces are modelled as uniform rods and they are smoothly hinged at B. The floor is smooth and the distance AC = 2.55m. The two metal parts are joined by a light horizontal rope of length 1.7m. The weight of AB is twice that BC. If the tension in the rope is 12.25N find W.

Firstly, we mark on a diagram the forces acting on the ladder as a whole:

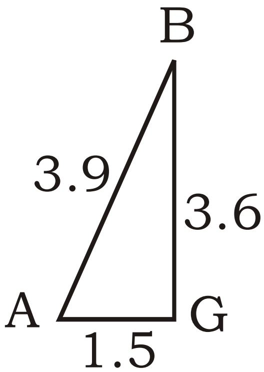


These are the weights of the two sections acting on their centres of mass situated half way up the ladder.

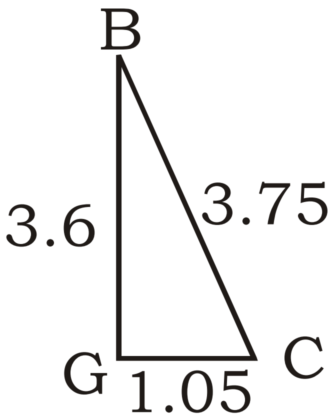
Resolution of the problem will depend on the geometry of the ladder, and hence a separate diagram for this is needed.



Since we will be taking moments experience tells us that we will need to know the perpendicular distances DE, EF, BE, EG, AG, GC so we proceed to calculate those using ratios and Pythagoras's theorem. If we know the perpendicular distances we will not, in fact, need to know the angles α and β. It is easier to work with perpendicular distances than with angles, so we use these wherever possible.



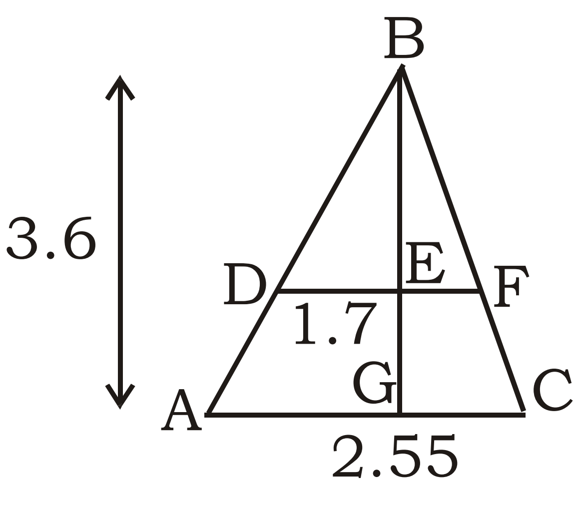
$AG = \sqrt{3.9^2 - 3.6^2} = 1.5$



$GC = \sqrt{3.75^2 - 3.6^2} = 1.05$

check AC = AG + GC

$2.55 = 1.5 + 1.05$



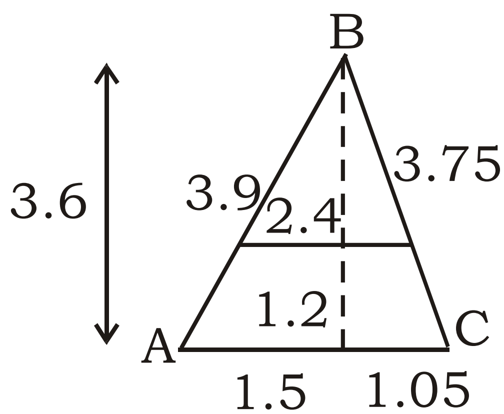
$$\frac{BE}{BG} = \frac{DF}{AC}$$

$$\therefore \frac{BE}{3.6} = \frac{1.7}{2.55}$$

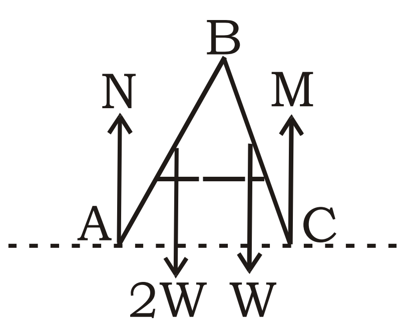
$$BE = \frac{1.7 \cdot 3.6}{2.55} = 2.4$$

$$\therefore EG = 1.2$$

The completed diagram is:



Recall that the external forces on the ladder are:



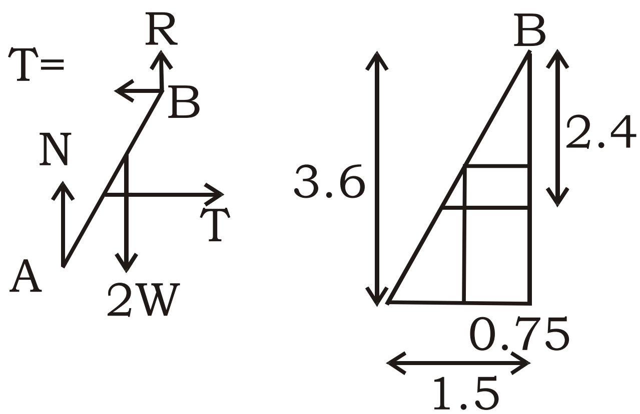
From this, by resolving vertically, we obtain the equation:

$$N + M = 2W + W$$

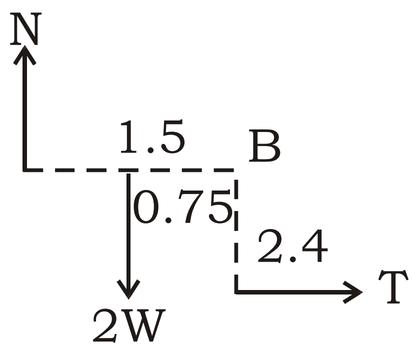
$$N + M = 3W \qquad (1)$$

This tells us that we seek equations involving N and M, since then we will be able to find W.

The forces acting on AB are:



Since the ladder is in static equilibrium the horizontal component of the reaction at B must equal the tension in the rope, T = 12.25. However, it is N we are interested in, so we choose to take moments at B.

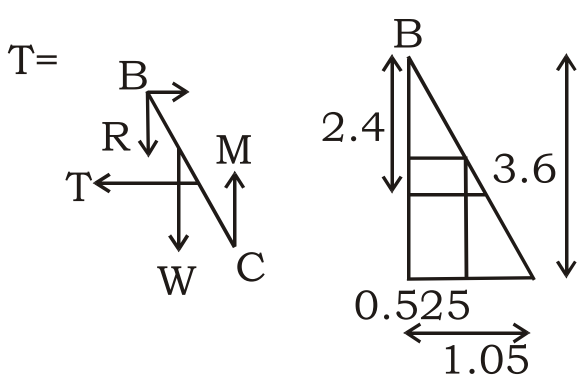


Taking moments at B eliminates the reaction R from the equation. Thus

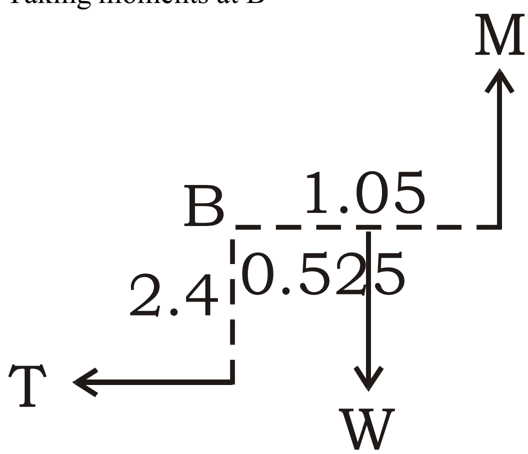
$1.5\text{N} = 1.5 \times 2\text{W} + 2.4 \times \text{T}$

Hence 1.5N = 1.5W + 29.4 (2)

The forces acting on BC are



Taking moments at B



$$1.05M = 0.525W + 2.4T$$

$$1.05M = 0.525W + 29.4 \qquad (3)$$

Hence, we have the three simultaneous equations:

$$N + M = 3W \qquad (1)$$

$$1.5N = 1.5W + 29.4 \qquad (2)$$

$$1.05M = 0.525W + 29.4 \qquad (3)$$

From (1): N = 3W – M

In (2)

1.5(3W - M) = 1.5W + 29.4

4.5W – 1.5M = 1.5W + 29.4

3W – 1.5M = 29.4 (4)

From (3) 0.525W – 1.05M = –29.4 (5)

$$(5)\times\frac{1.5}{1.05} \qquad 1.5W - 1.5M = -42 \qquad (6)$$

(4) – (6) 1.5W = 71.4

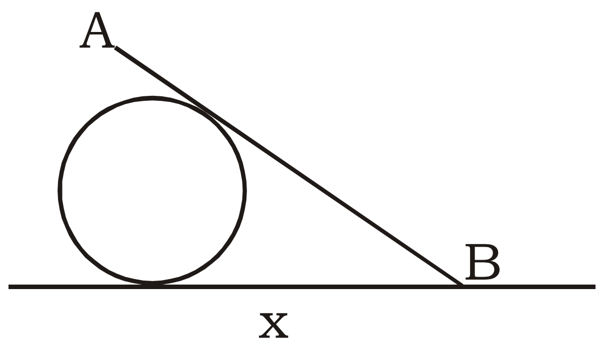
$\therefore$ W = 47.6N

This example illustrates a number of aspects of problem-solving in this area.

1. By resolving for the system as a whole and separately as parts we obtain a set of simultaneous equations, which are then solved algebraically.
2. The algebra can be simplified if moments are taken around appropriate points. However, provided moments are taken consistently the problem will be soluble. If you obtain a negative value for a force it merely means that you drew it in your diagram acting in the wrong direction.

We now proceed to a second example.

Example (2)



A uniform rod AB, of length 18a and mass 2m leans against a cylinder. Its base B is hinged to a rough floor. The whole system is in static equilibrium. there is no friction between the rod and the cylinder, but the contact between the cylinder and the floor is rough. The angle made by the rod with the floor is 2θ where

$\theta = \tanh^{-1}\left(5/12\right)$ .Should be tan

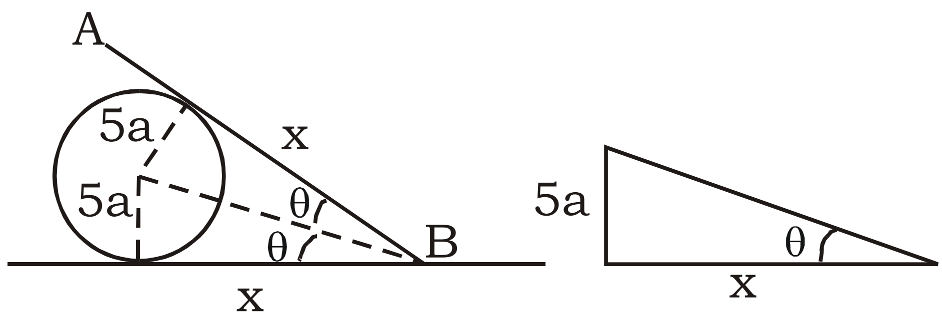
The cylinder has radius 5a and mass M.

(i) Find, in terms of m and g, the size of the reaction between the rod and the cylinder.

(ii) If the cylinder is just about to slip on the floor, show that the coefficient of friction between the cylinder and the floor is O.17 to 2 S.F.

Solution

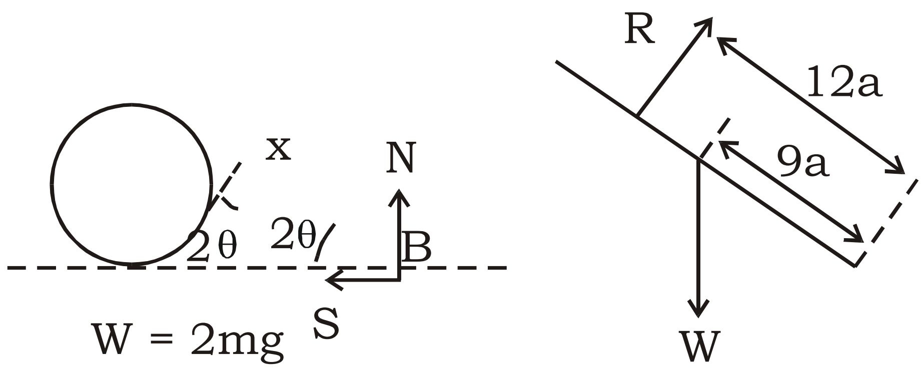
(i) The geometry of the problem is again important. We need to know the distance of the point of contact between the rod and the floor, B, and the point where the rod touches the cylinder.



We have

$$\tan\theta = \frac{5a}{x}$$
$$\therefore \frac{5}{12} = \frac{5a}{x}$$
$$\therefore x = 12a$$

The forces acting on the rod are as follows:



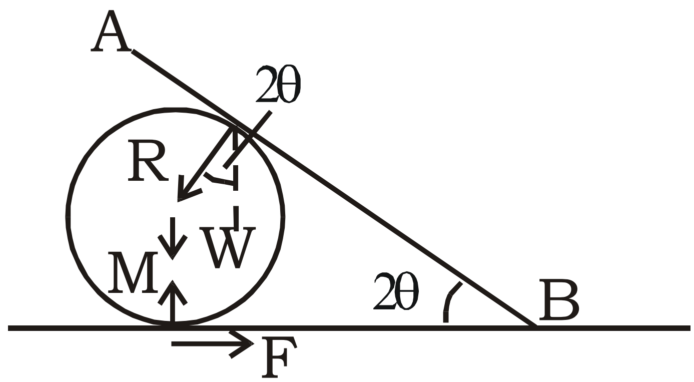
Here it makes sense to resolve using the angle 2θ. For this we use the trigonometric identity:

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \\ &= \frac{144-25}{169} \\ &= \frac{119}{169}\end{aligned}$$

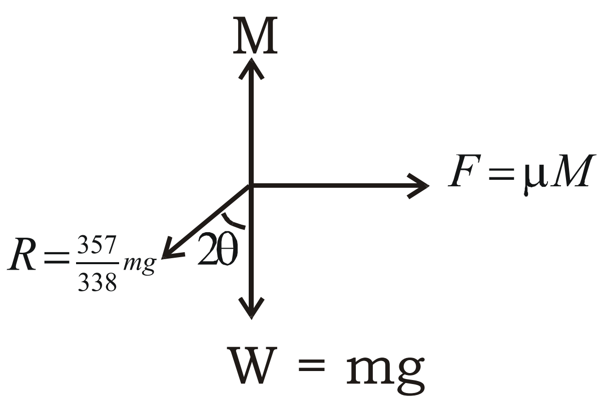
$$\therefore 12R = 18 \times \frac{119}{169} mg$$

$$R = \frac{357}{338} mg$$

(ii) The forces acting on the cylinder are:



Resolving the forces on the cylinder



$(\uparrow)$ $\quad W + R\cos 2\theta = M$

$$mg + \frac{357}{338} mg \times \frac{119}{169} = M$$

$$M = \frac{(57122 + 42483)mg}{338 \times 169} = \frac{99605}{338 \times 169} mg$$

$(\rightarrow)$ $\quad R\sin 2\theta = \mu M$

Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{5}{13}\times\frac{12}{13} = \frac{48}{169}$

$\therefore \frac{357}{338}mg\times\frac{48}{169} = \mu M$

Hence, $\mu = \frac{\mu M}{M}$

$$= \frac{\left(\frac{357\times 48}{338\times 169}\right)}{\left(\frac{99605}{338\times 169}\right)}$$

$= 0.172039...$

$= 0.17 \ (2.S.F.)$