Error in an interpolating polynomial

Theorem

Given the function $f(\mathrm{x})$ and an interpolating polynomial $p(\mathrm{x})$ of the degree $\leq n$ such that

$p(x_i) = f(x_i)$ for $i = 0,1\ldots n$

and $x_0 < x_1 < \ldots < x_n$

Then the error can be denoted by

$\varepsilon(\mathrm{x}) = p(x) - f(x)$

where x lies between x_0 and x_n and is given by

$$\varepsilon(\mathrm{x}) = -(x - x_0)(x - x_1)\ldots(x - x_n)\frac{f^{(n+1)}(C_x)}{(n+1)!}$$

where $\mathrm{C_x}$ is a number such that

$x_0 \leq C_x \leq x_n$

In this theorem we have

$$\varepsilon(\mathrm{x}) = -(x - x_0)(x - x_1)\ldots(x - x_n)\frac{f^{(n+1)}(C_x)}{(n+1)!}$$

The product $(x - x_0)(x - x_1)\ldots(x - x_n)$ ensures that

$\varepsilon(x_i) = 0$ whereever $p(x_i) = f(x_i)$

The expression $f^{(n+1)}$ means the $(n+1)$th derivertive of f.

This gives a prodedure for finding error bounds of interpolating polynomials.

1. You are given a function $f(\mathrm{x})$ and an interpolating polynomial $p(x)$ and points $x_{o,} x_1, \ldots x_n$, and the corresponding values $f(x_0), f(x_1), \ldots f(x_n)$.

2. Find the $(\mathrm{n+1})$th derivertiveof $f(x)$ and determine a value of M such that
$$\left|f^{(n+1)}(C)\right| \le M$$
for all the values of C between x_0 and x_n.

3. Then a bound for the error is
$$|\varepsilon(x)| \le \frac{|(x-x_0)(x-x_1)\ldots(x-x_n)|M}{(n+1)!}$$