Euler's and the Trapezoidal Method

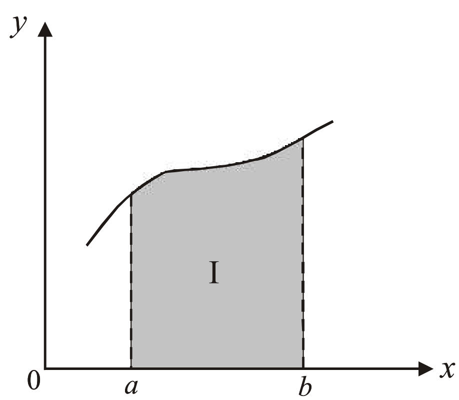
Numerical approximation

We wish to approximate the integral

$$I = \int_a^b f(x)\,dx$$

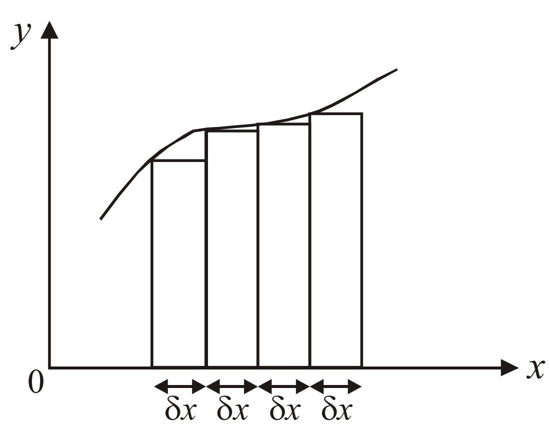
This represents the area under the graph of $f(x)$

It is a signed area - meaning that if the curve lies under the *x*-axis then the area will appear to be negative.

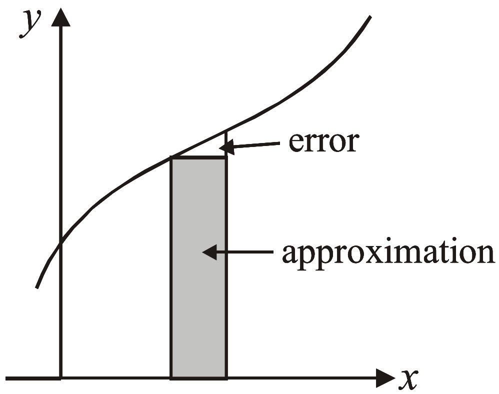


Euler's Method

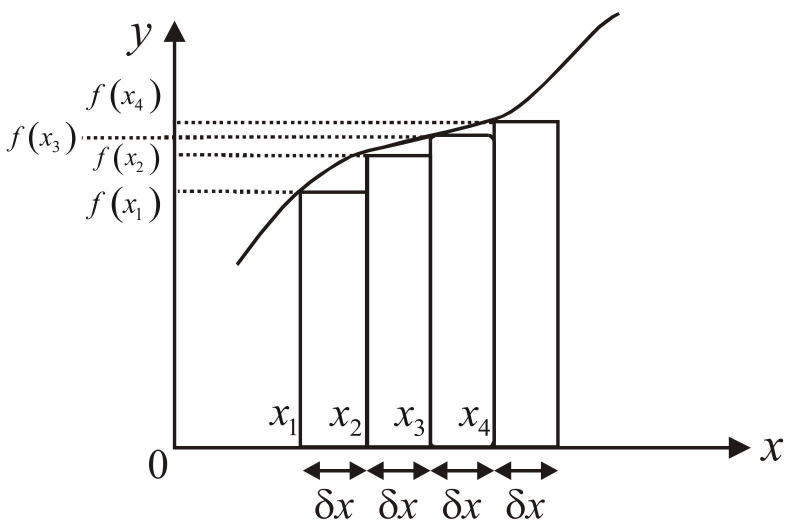
In Euler's method we approximate this area by series of rectangles each of some fixed width dx.



This is an approximation and consequently introduces an element of error- this being the sum of the areas between the curve and the rectangles



The approximation of the integral is



$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)\,dx \\
&\simeq \delta x\, f(x_0) + \delta x f(x_1) + \ldots + \delta x f(x_{n-1}) \\
&= \delta x \left\{ f(x_0) + f(x_1) + \ldots + f(x_{n-1}) \right\}
\end{aligned}
$$

Note that since $y_i = f(x_i)$ and we often use the symbol h or equivalent to represent the interval width, the rule for Euler's method can also be written

$$\int_a^b f(x)\,dx \ = \ h\left\{ y_0 + y_1 + \ldots + y_n \right\}$$

Example

Use Euler's method to determine the approximate value of

$$I = \int_2^3 \ln x dx$$

Use an interval width of $\delta x = 0.1$

Find the relative error in using this approximation

Solution

$$\int_a^b f(x)dx = \delta x\left(f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right)$$

$$\begin{aligned}\therefore \int_2^3 \ln x dx &= 0.1\left(\ln(2.0) + \ln(2.1) + \ldots + \ln(2.9)\right) \\ &= 0.1\left(0.6931... + 0.7419... + \ldots + 1.0647...\right) \\ &= 0.1(8.8913...) \\ &= 0.8891 \quad (4\ S.F)\end{aligned}$$

The exact value is given by

$$\begin{aligned}\int_2^3 \ln x dx &= \left[xlnx - x\right]_2^3 \\ &= (3\ln 3 - 3) - (2\ln 2 - 2) \\ &= 0.9095...\end{aligned}$$

$$\begin{aligned}\text{Percentage relative error} &= \frac{\text{error}}{\text{true value}} \times 100\% \\ &= \left(\frac{0.09095... - 0.8891...}{0.9095...}\right) \times 100\% \\ &= 22.5\% \quad \text{(3 S.F.)}\end{aligned}$$

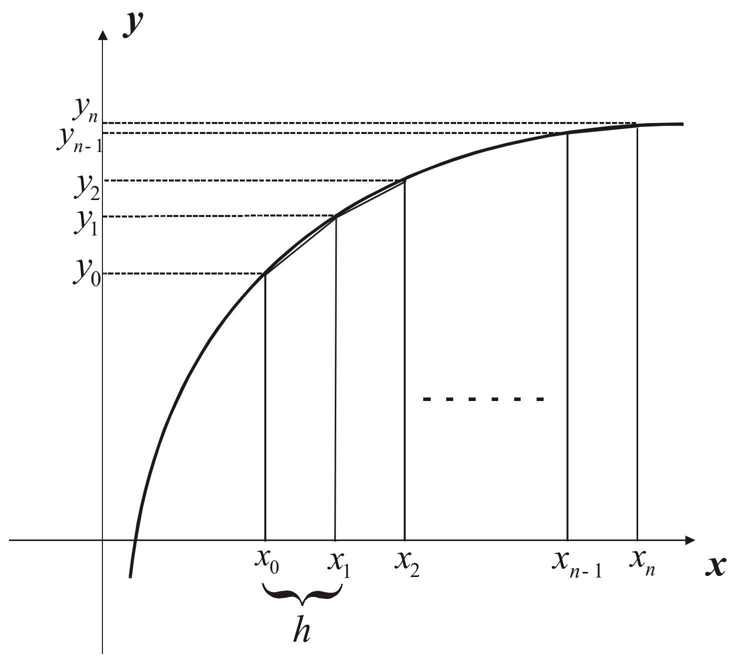
The trapezoidal method

As the above example shows, whenever we approximate an integral we introduce an error. The relative error in the above example was 22.5%, which would almost certainly be too large for any practical application. Thus, we wish to make the error involved in the approximation as small as possible. One way to do this is to make the interval width very small, but this is costly in computational time, which was a particular problem before the introduction of modern computers. Even now we would still want the most efficient computational process (algorithm). So another approach is to find a method that leaves out less of the area, and so, for a given interval width makes the error smaller.

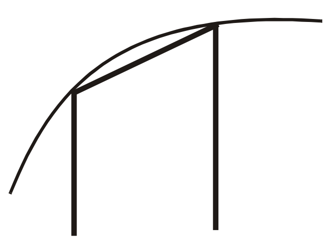
One such approach is to approximate the areas under the function by a series of trapezia. To find

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$

We approximate the area by trapezia.



Each trapezium is fitting just under the curve in this example:



This shows that in this case the approximation is an under-estimate. However, it also shows that the area left out of the approximation is much smaller than the corresponding area for Euler's method.

The width of each trapezium is h.

The area of each trapezium is $\frac{h}{2}\left(y_k + y_{k+1}\right)$

When all the trapezia are added together, the middle ordinates (the y_k values) appear twice and the end ordinates (y_1 and y_n) appear twice.

$$A \approx \frac{h}{2}\left(y_0 + y_1\right) + \frac{h}{2}\left(y_1 + y_2\right) + \ldots + \frac{h}{2}\left(y_{n-1} + y_n\right)$$

$$\therefore \int_a^b f\left(x\right)dx \approx \frac{h}{2}\left\{\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right\}$$

Where $h = \dfrac{b-a}{n}$

Of course, you should in fact already be aware of the trapezoidal method, which is also called the trapezium method. It is usual to *introduce* the idea of integration by introducing the trapezium method *first*. The trapezium method is shown as a method of find an approximation to an *area* under a curve *before* this area is equated with an integral. What this means is that all of this should be revision. In fact, the only new idea is that of expressing the "area" by the integral symbol. Instead of

Area $\frac{h}{2}\left\{\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right\}$

we write

$$I \approx \int_a^b f\left(x\right)dx \approx \frac{h}{2}\left\{\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right\}$$

However, this introduces a second point about the trapezoidal method. Strictly speaking the concept of an integral is of much greater generality than the concept of an area. Integrals of functions find the areas under them in their corresponding graph, but not all integrals should be interpreted as areas. In moving from "area" to $\int$ we have also taken a step towards greater abstraction and generality.

Example

Use the trapezoidal method to determine the approximate value of

$$I = \int_2^3 \ln x dx$$

Use an interval width of $h = \delta x = 0.1$

Find the relative error in using this approximation

Solution

$$I \approx \int_a^b f(x)\,dx \approx \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{0.1}{2}\{(\ln(2.0) + \ln(3.0)) + 2(\ln(2.1) + \ln(2.2) + \ldots + \ln(2.9))\} \\ &= 0.0.5\{(0.6931... + 1.0986...) + 2(0.7419... + 0.7884... + \quad .... \quad + 1.0647...)\} \\ &= 0.90940364...\end{aligned}$$

The exact value is given by

$$\begin{aligned}\int_2^3 \ln x dx &= [xlnx - x]_2^3 \\ &= (3\ln 3 - 3) - (2\ln 2 - 2) \\ &= 0.909542504...\end{aligned}$$

$$\begin{aligned}\text{Percentage relative error} &= \frac{\text{error}}{\text{true value}} \times 100\% \\ &= \left(\frac{0.0909542504... - 0.90940364...}{0.9095...}\right) \times 100\% \\ &= 0.0117\% \ (3\ S.F.)\end{aligned}$$

This illustrates the superiority of the trapezoidal method over Euler's method.