Euler's formula

The exponential version of a complex number

When we multiply complex numbers in polar (also called trigonometric) form we multiply the moduli and add the arguments

$$[r_1,\theta_1][r_2,\theta_2] = [r_1r_2,\theta_1+\theta_2]$$

This process of adding the arguments, instead of multiplying them, is suggestive of the situation with exponentials. Recall that when two numbers to the same base are multiplied, then we add the exponents

$$a^x a^y = a^{x+y}$$

This suggests that it should be possible to represent the process of multiplying complex numbers in polar form in a similar way using an exponent notation. This is in fact true. If $z_1 = [r_1,\theta_1]$ and $z_2 = [r_2,\theta_2]$, then z_1 and z_2 can be represented as

$$z_1 = r_1e^{i\theta_1}, \quad z_2 = r_2e^{i\theta_2}$$

and their product is

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_2+\theta_2)}$$

This expresses in *exponential form* the central idea of multiplication of complex numbers in polar form, which is to multiply the moduli and add the arguments.

Euler's formula

The possibility of this exponent representation depends on a result known as Euler's formula.

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

This is proven using Maclauren's theorem as follows. The proof will require knowledge of the following standard series expansions, which are derived from Maclauren's theorem.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Proof

Let $z = x + iy$, then

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x \left\{ 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots \right\} \\ &= e^x \left\{ 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - \ldots \right\} \quad \text{since } i^2 = -1 \\ &= e^x \left\{ \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots \right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots \right) \right\} \\ &= e^x (\cos y + i \sin y) \end{aligned}$$

Euler's formula enables us to represent a complex number z as the image of another complex number w, to which w is related by the exponent function.

$$z = e^w$$

The relationship is such that the modulus of z is the exponent of the real part of w, and the argument of z is the imaginary part of w.

$$|z| = e^{\text{Re}(w)}$$
$$\arg z = \text{Im}(w)$$

The polar (trigonometric) representation respectively of a number z is

$$z = [r, \theta] = r(\cos\theta + i \sin\theta)$$

to which we can add the exponent form

$$z = re^{i\theta}$$

where r is the modulus of z and θ is the argument of z.

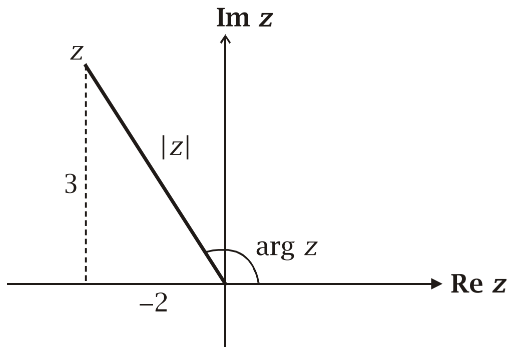
Example (1)

Express $z = -2 + 3i$ in the form $re^{i\theta}$.

Solution

$$r = |z| = \sqrt{2^2 + 3^2} = \sqrt{13}$$
$$\theta = \arg z = \tan^{-1}\left(3/_{-2}\right) = 2.16 \quad (3.S.F.)$$
$$z = re^{i\theta} = \sqrt{13}e^{2.16i}$$

The following sketch illustrates this solution.



Example (2)

Express

$4e^{i\frac{3\pi}{4}}$ in the form $a+ib$.

Solution

$$4e^{i\frac{3\pi}{4}} = 4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$= -2.83 + 2.83i \quad (2 \text{ d.p.})$$