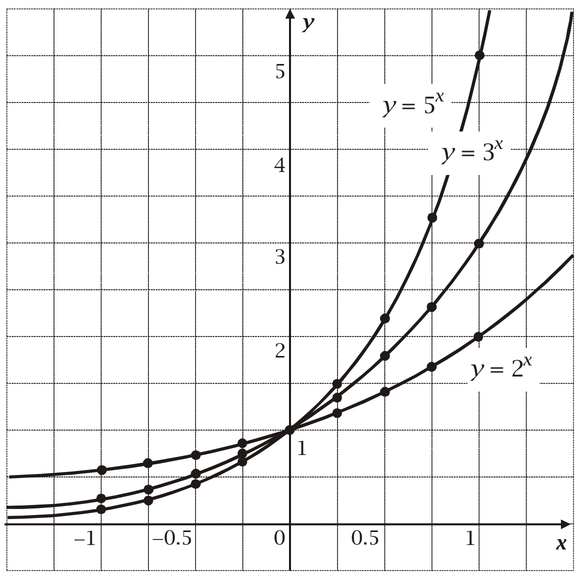
Exponential and Natural Logarithm

Prerequisites

This chapter builds on you knowledge of exponential functions and logarithms. Consider the graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$ for values $-1 < x < 1$.



These are all curves corresponding to exponential functions of the form $y = a^x$.

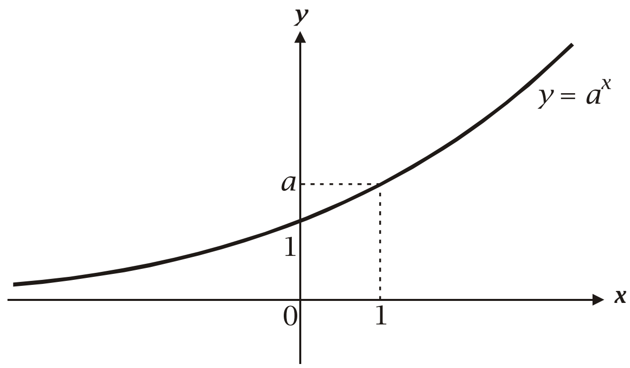
This may also be written

$y = \exp_a x$.

This makes it clear that y is the value of a function that we call exponent, which depends on *two* arguments, both of which may be any real number positive or negative – a variable x and a base a. Functions of the form $y = a^x$ comprise a family of functions, which are all similar in certain ways. All the curves have the same basic shape. They are always increasing. The curves are asymptotic to the negative x-axis. They pass through the point $(1, a)$. They all pass through the point $(0,1)$.

The curves differ from each other only in their degree of steepness: $y = 5^x$ is steeper than $y = 3^x$ which is steeper than $y = 2^x$.

It is usual to regard the number a, called the base, as fixed, and the number x as variable. Exponent, for a given base, is a function of the variable x.



The general graph of the exponential function $y = a^x$

Transformation of graphs of exponential functions

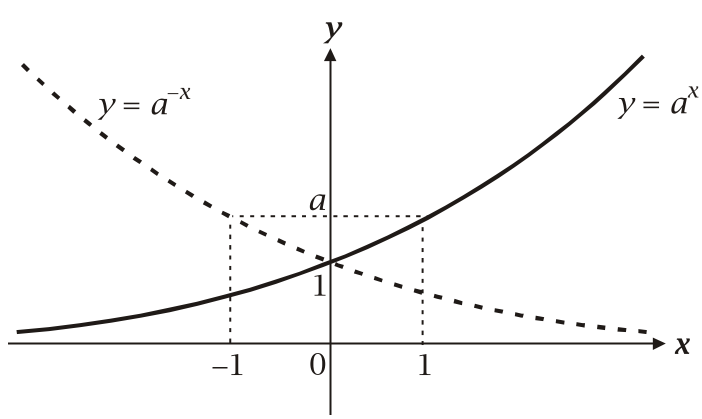
You should also already be familiar with transformation of graphs by horizontal and vertical translations and scalings, including reflections of graphs in the x and y-axes.

Example (1)

Use your knowledge of transformation of graphs to sketch $y = a^{-x}$.

Solution

$y = a^{-x}$ is the reflection of $y = a^x$ in the y-axis



Example (2)

(*a*) On the same graph sketch $(1)\ y = 2^x, (2)\ y = 2^{x-2}$ and $(3)\ y = -2^{x-2}$

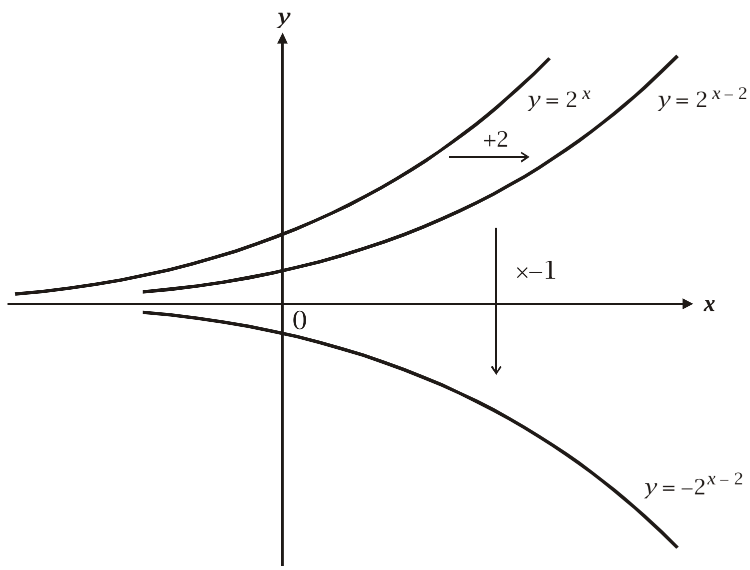
(*b*) How may the graph of $y = -2^{x-2}$ be obtained from the graph of $y = 2^x$ by a series of transformations?

Solution

(*a*) Graphs of (1) $y = 2^x$

(2) $y = 2^{x-2}$

(3) $y = -2^{x-2}$



(*b*) $y = 2^{x-2}$ is obtained from $y = 2^x$ by a horizontal translation

$x \to x + 2$

$y = -2^{x-2}$ is obtained from $y = 2^{x-2}$ by a reflection in the *x*-axis

$y \to -y$

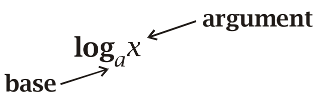
Logarithm

The logarithm is the inverse of the exponential function. It is written

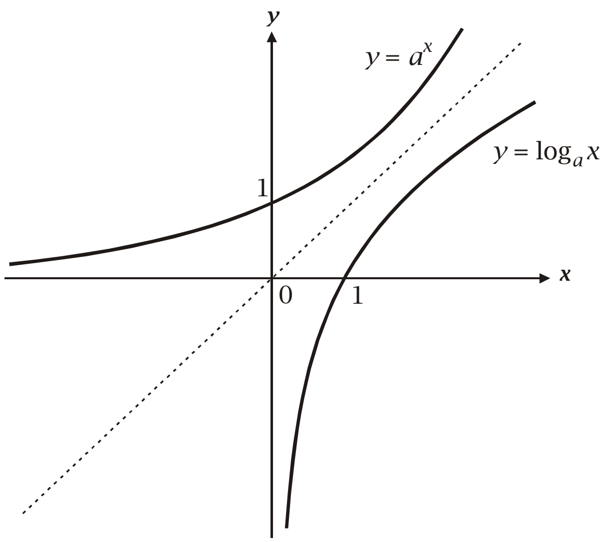
$y = \log_a x$

This is read "log to the base *a*, *x*".

Like the exponential function logarithm depends on two numbers, the base and exponent.



The inverse of a function is the reflection of that function in the line $y = x$.



The logarithmic function $y = \log_a x$ like the exponential function $y = a^x$ is an always-increasing function. However, whereas $y = a^x$ gets steeper and steeper, the rate of increase of $y = \log_a x$ gets less and less. Nonetheless, it is always going up. It passes through the point $(1,0)$ on the *x*-axis, and is asymptotic to the negative *y*-axis. It is undefined for negative values of *x*, so the domain is the positive real line $x > 0$, and 0 is *not* included in the domain.

A special exponential function

A special exponential function is

$y = e^x$.

Here, the symbol *e* stands for a special number, *e* = 2.7182818….

The dots indicate that the decimal expansion of *e* carries on indefinitely. It is an irrational number – which means that it has an infinite, non-repeating decimal expansion.[1] The importance

[1] The definition and evaluation of the number *e* is beyond the level of this chapter. Here it may be taken to be the number such that $\frac{d}{dx}e^x = e^x$.

of e derives from the fact that the gradient of the tangent to $y = e^x$ at x is equal to the value of $y = e^x$ at x. This is written $\frac{d}{dx}e^x = e^x$.

The gradient of the tangent to $y = e^x$ at any given point is the same as its derivative.[2] So the expression $\frac{d}{dx}e^x = e^x$ also says that the derivative of $y = e^x$ at x is equal to the gradient of $y = e^x$ at x. Because of this property $y = e^x$ is regarded as the fundamental exponential function. Consider the following table

Operation	**Objects**	**Zero operation**	**Example**	**Explanation**
Addition	Numbers	Adding zero	$5 + 0 = 5$	Adding 0 to any number does not change that number
Multiplication	Numbers	Multiplying by 1	$5 \times 1 = 5$	Multiplying any number by 1 does not change that number

The property

$$\frac{d}{dx}e^x = e^x$$

makes $y = e^x$ a special function with regard to differentiation in a way that is similar to how zero is special to addition and 1 is special to multiplication. The analogy is not exact, but it may help you to appreciate the importance of the function $y = e^x$ and its role in differentiation.

Natural logarithm

The inverse of fundamental exponential function $y = e^x$ is called the *natural logarithm*. It is denoted by

$$y = \log_e x$$

or

$$y = \ln x.$$

The second of these forms is read, "y equals lin x". Whilst the expression $y = \log_e x$ makes it clear that the natural logarithm is taken to base e, we will adopt the symbolism $y = \ln x$ throughout our course. The following mapping diagram makes the relationship between $y = e^x$ and $y = \ln x$ clear.

[2] Proof of this result is beyond the level of this chapter.

$$y = e^{x} \underset{\exp_e}{\overset{\log_e}{\rightleftarrows}} x = \ln y$$

From this we may write

$$x = e^{\ln x}$$
$$x = \ln\left(e^{x}\right)$$

both of which follow from the definition of $y = \ln x$ as the inverse of $y = e^{x}$. Since these functions are so important in mathematics calculators have buttons for them, so evaluating a number to the power *e* or a natural logarithm poses no practical difficulties.

Example (3)

In this question give any numerical answers to 4 significant figures.

(*a*) Use a calculator to evaluate

(*i*) e^{4}

(*ii*) $e^{-0.3}$

(*iii*) $\ln 4$

(*iv*) $\ln(0.3)$

(*b*) Why is it not possible to evaluate $\ln(-0.3)$?

(*c*) Solve the equations

(*i*) $2 = e^{x}$

(*ii*) $-2 = \ln x$

(*d*) Why is it not possible to solve the equation $e^{x} = -0.3$?

Solution

(*a*) (*i*) $e^{4} = 54.598... = 54.60 \ (4 \text{ s.f.})$

(*ii*) $e^{-0.3} = 0.740818... = 0.7408 \ (4 \text{ s.f.})$

(*iii*) $\ln 4 = 1.386294... = 1.3863 \ (4 \text{ s.f.})$

(*iv*) $\ln(0.3) = -1.203972... = 1.204 \ (4 \text{ s.f.})$

(*b*) Logarithm is undefined for a negative number.

(*c*) (*i*) $2 = e^{x}$

$x = \ln 2 = 0.693147... = 0.6931 \ (4 \text{ s.f.})$

(*ii*) $-2 = \ln x$

$x = e^{-2} = 0.1353352... = 0.1353 \ (4 \text{ s.f.})$

(*d*) The equation $e^x = -0.3$ cannot be solved because $y = e^x$ is a positive real number $y > 0$ for all arguments *x*. Likewise, the equation $e^x = -0.3$ leads to $x = \ln(-0.3)$ and logarithm is undefined for negative *x*.

Change of base

Since all exponential functions have the same basic shape any exponential function can be transformed into one of the others by a change of base. For instance, the function

$y = 2^x$

can be transformed into the function

$y = 3^{kx}$

by a change of base.

Let $y = 2^x$
Then, $\log_3 y = \log_3 2^x$
$\log_3 y = x \log_3 2$
$y = 3^{x \log_3 2}$

Since, $y = e^x$ is the fundamental exponential function, it makes sense in many cases to rewrite one exponential function in terms of e^x.

If $y = a^x$ then $y = e^{x \ln a}$

Example (4)

Use a calculator to write

$y = 7^x$

in the form

(*i*) $y = e^{x \ln a}$

(*ii*) $y = 2^{kx}$

For real numbers *k* and *a*, giving these to 4 significant figures.

Solution

(*i*) $y = 7^x$
$\ln y = \ln 7^x = x \ln 7$
$y = e^{x \ln 7} = e^{1.946x}$ (4 s.f.)

(*ii*) $y = 7^x$

$\log_2 y = \log_2 7^x = x\log_2 7$

$\log_2 7 = \frac{\ln 7}{\ln 2} = 2.80735...$

$\log_2 y = 2.80735...x$

$y = 2^{2.807x}$ (4 s.f.)

Derivative of the natural logarithm

The derivative of the natural logarithm[3] is give by

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

If $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$

Example (5)

Integrate the following functions with respect to *x*.

(*i*) x^{-2}

(*ii*) x^{-1}

(*iii*) x^0

(*iv*) x

(*v*) x^2

Solution

(*i*) $\int x^{-2}\,dx = -x^{-1} + c = -\frac{1}{x} + c$

(*ii*) $\int x^{-1}\,dx = \ln x + c$

(*iii*) $\int x^0 dx = \int 1\,dx = x + c$

(*iv*) $\int x\,dx = \frac{1}{2}x^2 + c$

(*v*) $\int x^2\,dx = \frac{1}{3}x^3 + c$

As the example shows

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

fills in the missing gap for $x^{-1} = \frac{1}{x}$ for the differentiation and integration of polynomial functions.

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[3] Proof of this result is beyond the level of this chapter.