Exponential Growth and Decay

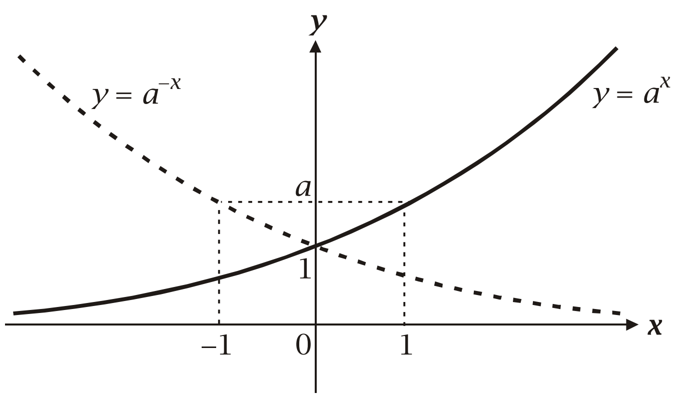
Prerequisites

(1) **Exponential functions**

You should be familiar with the properties of exponential functions. Recall that the exponential functions are a family of functions with the same general rule.

$x \to a^x$

This means multiply a by itself x times. This function depends on two numbers, a and x. Sometimes the function is written $\exp_a(x) = a^x$. It is usual to regard the number a, called the base, as fixed, and the number x as variable. The exponent function, for a given base, is a function of the variable x. The graphs of these functions all have essentially the same shape.



The general graph of the exponential functions $y = a^x$ and $y = a^{-x}$.

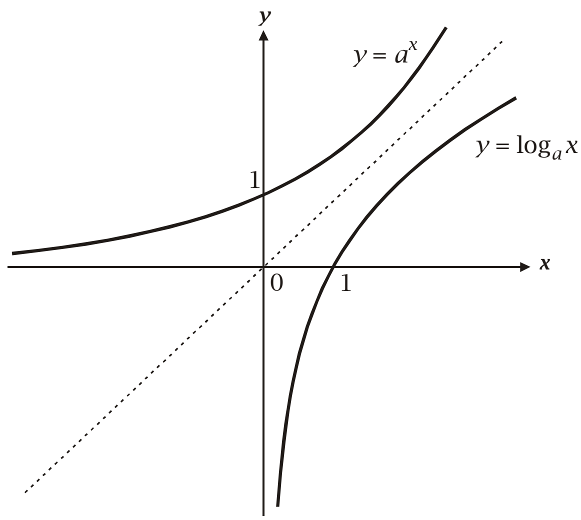
Exponential functions $y = a^x$ are a monotone increasing functions. Their graphs are asymptotic to the negative x –axis. They all pass through the point $(0,1)$. The curves differ from each other only in their degree of steepness: $y = 5^x$ is steeper than $y = 3^x$ which is steeper than $y = 2^x$. The exponential function $y = a^{-x}$ is the reflection of $y = a^x$ in the y-axis

(2) **Logarithm**

The inverse of the exponential function $y = a^x$ is logarithm, which is written

$y = \log_a x$

Logarithm depends on two numbers, the base a and the argument x. In the expression $y = \log x$ it is assumed that the base is 10. The graph of $y = \log_a x$ is the reflection of that function in the line $y = x$ of the graph of $y = a^x$.



The logarithmic function $y = \log_a x$ like the exponential function $y = a^x$ is an always-increasing function. The rate of increase of $y = \log_a x$ gets less and less. It passes through the point $(1,0)$ on the x-axis, and is asymptotic to the negative y-axis. It is undefined for negative values of x, so the domain is the positive real line $x > 0$, and 0 is *not* included in the domain.

(3) **Natural exponent**

A special exponential function is

$x \to e^x$

Here, the symbol e stands for a special number, e = 2.7182818…. This is an irrational number. The importance of e derives from the fact that the gradient of the tangent to $y = e^x$ at x is equal to the value of $y = e^x$ at x. This is written $\frac{d}{dx}e^x = e^x$.

(4) **Natural logarithm**

The inverse of fundamental exponential function $y = e^x$ is called the *natural logarithm*. It is denoted by

$y = \log_e x$ or $y = \ln x$.

The definition of the natural logarithm, $y = \ln x$, as the inverse of $y = e^x$ entails

$x = e^{\ln x} \quad \Leftrightarrow \quad x = \ln(e^x)$

(5) **Algebra of logarithms**

Logarithms are manipulated according the following rules

5.1 Addition

$$\log_a b + \log_a c = \log_a bc$$

5.2 Subtraction

$$\log_a b - \log_a c = \log_a \frac{b}{c}$$

5.3 Index

$$n \log_a b = \log_a b^n$$

5.1 Change of Base

$$\log_a b = \frac{\log_c b}{\log_c a}$$

(6) **Change the base of an exponent**

Since all exponential functions have the same basic shape any exponential function can be transformed into one of the others by a change of base. The function $y = a^x$ can be transformed into the function $y = b^{\ln ax}$. Since, $y = e^x$ is the fundamental exponential function, it makes sense in many cases to rewrite one exponential function in terms of e^x.

If $y = a^x$ then $y = e^{x \ln a}$

(7) **Derivatives**

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \log_a x = \frac{d}{dx}\left(\frac{1}{\ln a}\right) \ln x = \frac{1}{x \ln a}$$

Example (1)

Solve the equation

$$\frac{d}{dx} e^{2x} = 5e^x - 2$$

Solution

$$\frac{d}{dx} e^{2x} = 5e^x - 2$$

$$2e^{2x} - 5e^x + 2 = 0$$

Sub $u = e^x$

$2u^2 - 5u + 2 = 0$

$(2u-1)(u-2) = 0$

$u = \frac{1}{2}$ or $u = 2$

$e^x = \frac{1}{2}$ or $e^x = 2$

$x = \ln\left(\frac{1}{2}\right) = -0.693$ or $x = \ln 2 = 0.693$ (3 s.f.)

Modelling real life situations by exponential functions

In a certain quiz game the prize money is doubled each time the contestant gets an answer right; the contestant stops as soon as she gets an answer wrong. She starts with £1. Assuming that the contestant gets every question right, her winnings are an example of exponential growth. The contestant's winnings can be tabulated as follows.

No. of right questions	0	1	2	3	…..	n
Winnings	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$	….	2^n

In this example it does not make sense to talk of fractions of a right question – at the end of the contest she has a whole number of right questions. Her winnings are *modelled* by the exponential growth function

$\text{winnings}(£) = 2^n$ for n as an integer

The term *model* is used here when we apply a mathematical structure to a real life situation. In this case the contestant's winnings (real life property) are mirrored by a mathematical function. This predicts how much the contestant will win as a function of the number of questions is a row that she gets right.

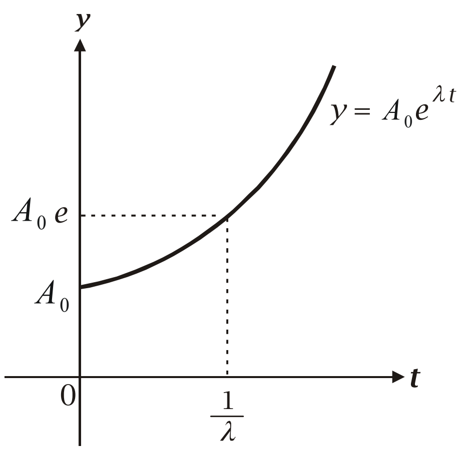
In other cases it is useful to model a real life situation by an exponential function with rational or real indices. For example, when bacteria grow in an environment that places no limitations on their population size, their population exhibits exponential growth. If the population doubles every 10 hours, it is still sensible to ask what the population is at intermediate times, for example, after 12.3 hours.

Exponential growth

An exponential growth function has the form

$y = A_0 e^{\lambda t}$

where A_0 is the inital value and $\lambda > 0$ is a constant.



Graph of an exponential growth function

An example of exponential growth is "every ten years the population doubles". This has equation $P = P_0 2^{\frac{t}{10}}$ where P_0 is the initial population. By a change of base this is also $P = P_0 e^{\frac{t}{10}\ln 2}$.

Example (2)

The size P of the population of a habitation is modelled as a continuous function of the real variable t, where t is in years.

$P = P_0 e^{kt}$

The initial size of the population at the foundation of the habitation is 10. At 100 years the population is $P_1 = 99$.

(*a*) Determine P_0 and k in $P = P_0 e^{kt}$. Give your answer for k as an exact logarithm.

(*b*) The habitation may be defined to be a metropolis when the population reaches 1,000,000 inhabitants. In what year after the foundation will the habitation become a metropolis?

Solution

(*a*) We are given $P_0 = 10$ therefore we have $P = 10e^{kt}$

When $t = 100$ we have $P = P_1 = 99$; therefore, on substituting into $P = 10e^{kt}$

$$99 = 10e^{100k}$$
$$e^{100k} = 9.9$$
$$100k = \ln(9.9)$$
$$k = \frac{1}{100}\ln(9.9)$$

(*b*) $$10^6 = 10e^{\left(\frac{1}{100}\ln(9.9)\right)t}$$
$$10^5 = e^{\left(\frac{1}{100}\ln(9.9)\right)t}$$
$$\ln(10^5) = \left(\frac{1}{100}\ln(9.9)\right)t$$
$$t = 100 \times \frac{\ln(10^5)}{\ln(9.9)} = 502.19\ldots$$

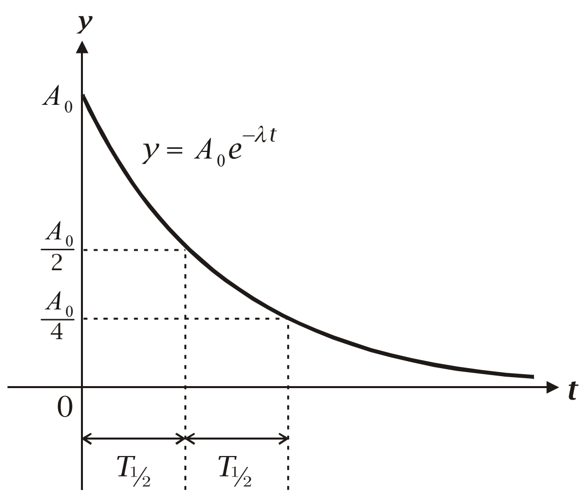
The answer is the next whole integer up, $t = 503$ years. That is, in the 503^{rd} year after the foundation.

Exponential decay

An exponential decay function has the form

$y = A_0 e^{-\lambda t}$

where A_0 is the initial value and $\lambda > 0$ is a constant. This constant is called the *decay constant*.



Graph of an exponential decay function

As the graph indicates a decay function has a *half-life*, which is the time period after which the value of the function is halved. The half-life is denoted by the symbol $T_{1/2}$. An example of a decay function is "every 5730 years the quantity of radioactive carbon-14 is halved." The half-life is $T_{1/2} = 5730$. The relationship between the half-life and the decay constant is

$$T_{1/2} = \frac{\ln 2}{\lambda}$$

Example (3)

Given $y = A_0 e^{-\lambda t}$ and the definition of half-life, $T_{1/2}$, as the time taken for the value of this function to be halved, prove $\lambda = \frac{\ln 2}{T_{1/2}}$ and $T_{1/2} = \frac{\ln 2}{\lambda}$.

Solution

Let $A = A_0 e^{-\lambda t}$

When $t = T_{1/2}$ we have $A = \frac{A_0}{2}$. Hence

$$\frac{A_0}{2} = A_0 e^{-\lambda T_{1/2}}$$

$$\frac{1}{e^{-\lambda T_{1/2}}} = 2$$

$$e^{\lambda T_{1/2}} = 2$$

$$\lambda T_{1/2} = \ln 2$$

$$\lambda = \frac{\ln 2}{T_{1/2}} \quad \text{and} \quad T_{1/2} = \frac{\ln 2}{\lambda}$$

Example (4)

When a certain quantity of hot water cools it is thought to modelled by an exponential decay function of the form

$$\theta = Ae^{-0.02t}$$

where θ is the temperature above room temperature, A is a constant and t is the time in minutes. A room has a constant temperature of 25°C and the temperature of a water bath is 75°C when $t = 0$. Find the temperature of the water bath when $t = 20$.

Solution

$$\theta = Ae^{-0.02t}$$

$$t = 0 \qquad \Rightarrow \qquad \theta = 75 - 25 = 50$$

$$50 = Ae^{0}$$

$$A = 50$$

$$t = 10 \qquad \Rightarrow \qquad \theta = 50e^{-0.02\times 10} = 50e^{-0.2} = 40.9°C$$

So the temperature of the water bath is $25 + 40.9 = 65.9°C$ (nearest 0.1°)

Example (5)

A radioactive source is placed at a distance from a Geiger counter. At time t = 0, the count is 1000; at time t = 4s, the count is 690. Find the decay constant and the half-life for this radioactive substance.

Solution

Let the count be A, then the decay has the form

$A = A_0 e^{-\lambda t}$

Since $A = 1000$ when $t = 0$, $A_0 = 1000$, and $A = 1000e^{-\lambda t}$

Also, $A = 690$ when $t = 4$

Hence

$690 = 1000e^{-4\lambda}$

$e^{-4\lambda} = \frac{690}{1000} = 0.690$

$-4\lambda = \ln 0.690$

$\lambda = 0.0928 \ (2 \text{ s.f.})$

$$\begin{aligned} T_{1/2} &= \frac{\ln 2}{\lambda} \\ &= \frac{\ln 2}{0.0928} \\ &= 7.47 \text{ s} \ (2 \text{ s.f.}) \end{aligned}$$