Summation of finite series using the method of differences

Prerequisites

You should be familiar with the summation notation and the method of summing finite series using standard results. We remind you, however, that a series is an expression of the form

$$\sum_{r=1}^{n} u_r$$

For example, let $u_r = \dfrac{1}{r(r+1)}$ then

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{k(k+1)} + \ldots + \frac{1}{n(n+1)}$$

There is no general method for summing every series, and consequently the student must learn a number of approaches that might be suitable. One such approach is the method of differences. Another prerequisite is that you should be familiar with the technique of partial fractions, which is used for spitting an algebraic faction with a problematic denominator into factions with simpler denominators. For example

$$\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$$

is found by letting

$$\frac{1}{r(r+1)} \equiv \frac{A}{r} + \frac{B}{r+1} \equiv \frac{A(r+1) + Br}{r(r+1)}$$

and solving for A and B by equating coefficients or by the cover-up method. You should be comfortable with this technique before commencing this chapter.

The method of differences

The method of differences is best shown through example. To introduce this method we will first show that

$$\sum_{r=1}^{4} \frac{1}{r(r+1)} = \frac{4}{5}$$

The method works by splitting the awkward general term in the series into two simpler terms. Here, by the method of partial fractions

$$\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$$

Hence

$$\begin{aligned}
\frac{1}{r(r+1)} &= \sum_{r=1}^{4}\left(\frac{1}{r} - \frac{1}{r+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} \\
&= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{4}\right) - \frac{1}{5} \\
&= 1 - \frac{1}{5} \\
&= \frac{4}{5}
\end{aligned}$$

In this example we see that all the middle terms cancel in pairs and the result is obtained by subtracting the last term from the first. We also see that this would have been the case had we summed

$$\sum_{r=1}^{5}\frac{1}{r(r+1)}$$

or

$$\sum_{r=1}^{6}\frac{1}{r(r+1)}$$

or any finite series with the same form where $u_r = \frac{1}{r(r+1)}$

We can immediately sum both these series

$$\sum_{r=1}^{5}\frac{1}{r(r+1)} = \sum_{r=1}^{5}\left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{1} - \frac{1}{6} = \frac{5}{6}$$

or

$$\sum_{r=1}^{6}\frac{1}{r(r+1)} = \sum_{r=1}^{6}\left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{1} - \frac{1}{7} = \frac{6}{7}$$

So whenever terms can be made to cancel in pairs there is a method that provides a short cut to the sum of a series. This method applies whenever the r th term, u_r, can be expressed as

$$u_r \equiv f(r+1) - f(r)$$

where $f(r)$ is a function of r.

Hence, if

$$u_r \equiv f(r+1) - f(r)$$

then

$$\sum_{r=1}^{n} u_r = f(n+1) - f(1)$$

This can be proven as follows

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left(f(r+1) - f(r)\right)$$
$$= \left(f(n+1) - f(n)\right) + \left(f(n) - f(n-1)\right) + \ldots + \left(f(2) - f(1)\right)$$
$$= f(n+1) + \left(f(n) - f(n)\right) + \left(f(n-1) - f(n-1)\right) + \ldots + \left(f(2) - f(2)\right) - f(1)$$
$$= f(n+1) - f(1)$$

The method works the other way around. That is, if $u_r \equiv f(r) - f(r+1)$ then

$$\sum_{r=1}^{n} u_r = f(1) - f(n+1)$$

Example

Using the method of differences, find

$$\sum_{r=1}^{n} \frac{3}{(3r-2)(3r+1)} = \frac{3}{1\times 4} + \frac{3}{4\times 7} + \frac{3}{7\times 10} + \ldots$$

Solution

We must begin by splitting the general term

$$\frac{3}{(3r-2)(3r+1)}$$

into partial fractions as follows

$$\frac{3}{(3r-2)(3r+1)} = \frac{A}{3r-2} + \frac{B}{3r+1} = \frac{A(3r+1) + B(3r-2)}{(3r-2)(3r+1)}$$

Equating coefficients

$$A(3r+1) + B(3r-2) = 3$$

$A = -B$ $\quad$ $A - 2B = 3A = 3$ $\quad$ $A = 1$ $\quad$ $B = -A = -1$

$$\therefore \frac{3}{(3r-2)(3r+1)} = \frac{1}{3r-2} - \frac{1}{3r+1}$$

Hence

$$\sum_{r=1}^{n} \frac{3}{(3r-2)(3r+1)} = \frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \ldots + \frac{1}{3n-2} - \frac{1}{3n+1}$$
$$= 1 - \frac{1}{3n+1}$$
$$= \frac{3n}{3n+1}$$

Note, here we have displayed the process in full, but the method allows us in this case to jump directly to the line

$$\sum_{r=1}^{n} \frac{3}{(3r-2)(3r+1)} = 1 - \frac{1}{3n+1}$$

This is because the result

$$\frac{3}{(3r-2)(3r+1)} = \frac{1}{3r-2} - \frac{1}{3r+1}$$

puts the sum into the form

$$\sum_{r=1}^{n} \frac{3}{(3r-2)(3r+1)} = \sum_{r=1}^{n} f(r) - f(r+1)$$

where

$$f(r) = \frac{1}{3r-2}$$

and

$$f(r+1) = \frac{1}{3(r+1)-2} = \frac{1}{3r+1}.$$

So we are allowed to “jump” immediately to the result

$$\sum_{r=1}^{n} \frac{3}{(3r-2)(3r+1)} = f(1) - f(n+1)$$

$$= 1 - \frac{1}{3n+1}$$

Note, we can “tidy-up” this last expression a little to obtain

$$1 - \frac{1}{3n+1} = \frac{3n+1-1}{3n+1} = \frac{3n}{3n+1}$$

Either expression would do as a final solution to the problem.