First Isomorphism Theorem for Groups

Quotient groups and the first isomorphism theorem

Lemma, coset group

Let N be a normal subgroup of a group, G. Let *multiplication of cosets* of N be defined by $(Ng_1)(Ng_2) = N(g_1g_2)$

Let the set of cosets of N in G be denoted by $\frac{G}{N}$; Then

1. $\frac{G}{N}$ with the operation of multiplication of cosets is a group.
2. There exists a homomorphism

   $\phi : G \to \frac{G}{N}$ such that $\ker(\phi) = N$

$\frac{G}{N}$ is called the *quotient group* or *factor group* of G by N.

<u>Proof</u>

1. We must first show that multiplication of cosets is well defined.

   Let $Ng_1 = Ng_1'$ and $Ng_2 = Ng_2'$

   We must show $Ng_1g_2 = Ng_1'g_2'$.

   Now

   $Ng_1 = Ng_1' \Rightarrow g_1' = n_1g_1$

   $Ng_2 = Ng_2' \Rightarrow g_2' = n_2g_2$ where $n_1, n_2 \in N$

   $g_1'g_2' = n_1g_1n_2g_2$

   Since $N \lhd G$ then $g_1n_2g_1^{-1} \in N \Rightarrow g_1n_2g_1^{-1} = n_3 \in N \Rightarrow g_1n_2 = n_3g_1$

   Hence $g_1'g_2' = n_1n_3g_1g_2 \Rightarrow g_1'g_2' \in Ng_1g_2$ since $n_1n_3 \in N$. Hence $Ng_1g_2 = Ng_1'g_2'$ as required.

2. Now we must show that $\frac{G}{N}$ with the operation of multiplication of cosets is a group.

   This requires verification of the group axioms.

   2.1 Closure

Let Ng_1 and $Ng_2 \in \frac{G}{N}$, then $(Ng_1)(Ng_2) = Ng_1g_2 \in \frac{G}{N}$.

2.2 Identity

The identity in $\frac{G}{N}$ is N, for $(N)(Ng) = (N1)(Ng) = N1g = Ng$

2.3 Inverses

The inverse of (Ng) in $\frac{G}{N}$ is (Ng^{-1}) for $(Ng)(Ng^{-1}) = Ngg^{-1} = N$

2.4 Associativity

$$\begin{aligned}((Ng_1)(Ng_2))(Ng_3) &= (Ng_1g_2)(Ng_3) \\ &= N((g_1g_2)g_3) \\ &= N(g_1(g_2g_3)) \\ &= (Ng_1)(Ng_2g_3) \\ &= (Ng_1)((Ng_2)(Ng_3))\end{aligned}$$

3. Let

$$\phi\begin{cases} G \to \frac{G}{N} \\ g \mapsto gN \end{cases}$$

Then ϕ is a homomorphism since

$$\phi(g_1g_2) = Ng_1g_2 = (Ng_1)(Ng_2) = \phi(g_1)\phi(g_2)$$

Then

$$\begin{aligned}\ker(\phi) &= \{g \in G \mid \phi(g) = N\} \\ &= \{g \in G \mid Ng = N\} \\ &= \{g \in G \mid g \in N\} \\ &= N\end{aligned}$$

Definition, natural homomorphism

The homomorphism

$$\phi\begin{cases} G \to \frac{G}{N} \\ g \mapsto gN \end{cases}$$

is called the *natural homomorphism* from G onto $\frac{G}{N}$.

Result, quotient groups

1. Quotient groups of cyclic groups are cyclic.

2. Quotient groups of Abelian groups are Abelian.

Proof

This is a direct consequence of the result that homomorphisms map cyclic groups to cyclic groups, and Abelian groups to Abelian groups. Though this should be proven somewhere.

Theorem, kernel

The kernel of a group homomorphism $\phi : G \to H$ is a normal subgroup of G.

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Theorem, correspondence

Let $\phi : G \to H$ be a group homomorphism. Then there exists a one-one correspondence between cosets of $\ker(\phi)$ and elements of $\operatorname{Im}(\phi)$.

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First isomorphism theorem

Let $\phi : G \to H$ be a group homomorphism. Then $\frac{G}{\ker(\phi)} \cong \operatorname{Im}(\phi)$.

Proof

By the theorem above on the kernel of a group homomorphism, $\ker(\phi)$ is a normal subgroup of G. Then by the lemma above on cosets $\frac{G}{\ker(\phi)}$ is a group and there exists a homomorphism $\phi : G \to \frac{G}{\ker(\phi)}$. By the correspondence theorem above, there is a one-one correspondence ψ between cosets of $\ker(\phi)$ and $\operatorname{Im}(\phi)$. We have

$$\psi \begin{cases} \frac{G}{\ker(\phi)} \to \operatorname{Im}(\phi) \\ g\ker(\phi) \mapsto \phi(g) \end{cases}$$

Furthermore, ψ is a homomorphism, since

$$\psi(g_1 \ker(\phi))\psi(g_2 \ker(\phi)) = \phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \psi(g_1 g_2 \ker(\phi))$$

Hence ψ is an isomorphism and $\frac{G}{\ker(\phi)} \cong \operatorname{Im}(\phi)$.

Remark

The first isomorphism theorem establishes an association between normal subgroups, homomorphisms and quotient groups. Any normal subgrous is the kernel of a natural homomorphism. The image of this homomorphism is isomorphic to the quotient group.

Examples

1. the dihedral group D_8 we have the following permutations

permutation		**Symmetry of the square**		**sgn**
e	I	identity	even	+1
$(1,2)(3,4)$	Q_1	horizontal reflection	odd	–1
$(1,4)(2,3)$	Q_2	vertical reflection	odd	–1
$(2,4)$	Q_3	diagonal reflection	even	+1
$(1,3)$	Q_4	diagonal reflection	even	+1
$(1,2,3,4)$	$R_{\pi/2}$	rotation about the centre by $\pi/2$	odd	–1
$(1,3)(2,4)$	R_{π}	rotation about the centre by π	even	+1
$(4,3,2,1)$	$R_{3\pi/2}$	rotation about the centre by $\pi/2$	odd	–1

The map

$$\operatorname{sgn}\begin{cases} S_n \to S \\ f \mapsto \operatorname{sgn}(f) \end{cases}$$

is a group homomorphism. The kernel is the set of even permutations

$$\ker(\phi) = \{f \in D_8 \mid \operatorname{sgn}(f) = +1\} = \{I, Q_1.Q_2, R_\pi\}$$

This partitions D_8 as follows.

	I	Q_1	Q_2	R_π	$R_{\frac{\pi}{2}}$	$R_{\frac{3\pi}{2}}$	Q_3	Q_4
I	I	Q_1	Q_2	R_π	$R_{\frac{\pi}{2}}$	$R_{\frac{3\pi}{2}}$	Q_3	Q_4
Q_1	Q_1	I	R_π	Q_2	Q_4	Q_3	$R_{\frac{3\pi}{2}}$	$R_{\frac{\pi}{2}}$
Q_2	Q_2	R_π	I	Q_1	Q_3	Q_4	$R_{\frac{\pi}{2}}$	$R_{\frac{3\pi}{2}}$
R_π	R_π	Q_2	Q_1	I	$R_{\frac{3\pi}{2}}$	$R_{\frac{\pi}{2}}$	Q_4	Q_3
$R_{\frac{\pi}{2}}$	$R_{\frac{\pi}{2}}$	Q_3	Q_4	$R_{\frac{3\pi}{2}}$	R_π	I	Q_2	Q_1
$R_{\frac{3\pi}{2}}$	$R_{\frac{3\pi}{2}}$	Q_4	Q_3	$R_{\frac{\pi}{2}}$	I	R_π	Q_1	Q_2
Q_3	Q_3	$R_{\frac{3\pi}{2}}$	$R_{\frac{\pi}{2}}$	Q_4	Q_1	Q_2	I	R_π
Q_4	Q_4	$R_{\frac{\pi}{2}}$	$R_{\frac{3\pi}{2}}$	Q_3	Q_2	Q_1	R_π	I

The cosets of sgn are

$K = \ker(\phi) = \{I, Q_1.Q_2, R_\pi\}$

$K \cdot \frac{\pi}{2} = \left\{R_{\frac{\pi}{2}}, R_{\frac{3\pi}{2}}, Q_3, Q_4\right\}$

The image set is $\{+1,-1\}$, and sgn maps

$K \to +1 \qquad K \cdot \frac{\pi}{2} \to -1$

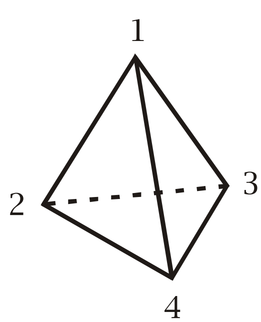
The quotient group has the following combination table and corresponding image.

	K	$K\frac{\pi}{2}$
K	K	$K\frac{\pi}{2}$
$K\frac{\pi}{2}$	$K\frac{\pi}{2}$	K

	$+1$	-1
$+1$	$+1$	-1
-1	-1	$+1$

This illustrates the idea that the groups $\frac{G}{K}$ and $\text{Im}(\text{sgn})$ are isomorphic.

2. The symmetry group S_4 comprises all symmetries of the tetrahedron. In the tetrahedron there are six edges, which come in three pairs of opposites. We may denote these opposite edges by the labels *A, B* and *C*. Numbering the vertices of the tetrahedron as follows



Then the edge pairs are

A edges $\{1,2\}$ and $\{3,4\}$

B edges $\{1,3\}$ and $\{2,4\}$

C edges $\{1,4\}$ and $\{2,3\}$

Let $g=(1,2,3)$, then g maps the edge $\{1,2\}$ to $\{2,3\}$ and the edge $\{3,4\}$ to $\{1,4\}$. That is, g maps the edge A to the edge C. Likewise $(1,2,3)$ maps C to B, and B to A. Hence the image of $g=(1,2,3)$ is the cycle (ABC).

Let ϕ denote the map from S_4 that arises from taking an element of $g \in S_4$ and finding the cycle to which g maps the edge pairs A, B and C.

$$\phi\begin{cases} S_4 \to S_3 \\ g \mapsto \text{A cycle of the symbols } A, B, C \end{cases}$$

We have just shown that

$\phi((1,2,3)) = (ABC)$.

It can be shown that ϕ is a group homomorphism with

$\ker(\phi) = \mathbf{V} = \{I,(12)(34),(13)(24),(14)(23)\}$

The following table gives the cosets of $\mathbf{V}$ and the images of the 24 permutations of S_4 under ϕ.

coset	g	$f(g)$
V	1	I
	$(12)(34)$	
	$(13)(24)$	
	$(14)(23)$	
$\mathbf{V}(123)$	(123)	(ACB)
	$(12)(34)(123)=(134)$	
	$(13)(24)(123)=(243)$	
	$(14)(23)(123)=(142)$	
$\mathbf{V}(132)$	(132)	(ABC)
	$(12)(34)(132)=(234)$	
	$(13)(24)(132)=(124)$	
	$(14)(23)(132)=(143)$	
$\mathbf{V}(12)$	(12)	(BC)
	$(12)(34)(12)=(34)$	
	$(13)(24)(12)=(1324)$	
	$(14)(23)(12)=(1423)$	
$\mathbf{V}(13)$	(13)	(AC)
	$(12)(34)(13)=(1234)$	
	$(13)(24)(13)=(24)$	
	$(14)(23)(13)=(1423)$	
$\mathbf{V}(14)$	(14)	(AB)
	$(12)(34)(14)=(1243)$	
	$(13)(24)(14)=(1342)$	
	$(14)(23)(14)=(23)$	

The quotient group $\frac{S_4}{\mathbf{V}}$ is isomorphic to S_3.

	I	(ABC)	(ACB)	(BC)	(AB)	(AC)
I	I	(ABC)	(ACB)	(BC)	(AB)	(AC)
(ABC)	(ABC)	(ACB)	I	$(1,3)$	(BC)	(AB)
(ACB)	(ACB)	I	(ABC)	(AB)	(AC)	(BC)
(BC)	(BC)	(AB)	(AC)	I	(ABC)	(ACB)
(AB)	(AB)	(AC)	(BC)	(ACB)	I	(ABC)
(AC)	(AC)	(BC)	(AB)	(ABC)	(ACB)	I

	I	$\mathbf{V}(132)$	$\mathbf{V}(123)$	$\mathbf{V}(12)$	$\mathbf{V}(14)$	$\mathbf{V}(13)$
I	I	$\mathbf{V}(132)$	$\mathbf{V}(123)$	$\mathbf{V}(12)$	$\mathbf{V}(14)$	$\mathbf{V}(13)$
$\mathbf{V}(132)$	$\mathbf{V}(132)$	(ACB)	I	$\mathbf{V}(1,3)$	$\mathbf{V}(12)$	$\mathbf{V}(14)$
$\mathbf{V}(123)$	$\mathbf{V}(123)$	I	$\mathbf{V}(132)$	$\mathbf{V}(14)$	$\mathbf{V}(13)$	$\mathbf{V}(12)$
$\mathbf{V}(12)$	$\mathbf{V}(12)$	$\mathbf{V}(14)$	$\mathbf{V}(13)$	I	$\mathbf{V}(132)$	$\mathbf{V}(123)$
$\mathbf{V}(14)$	$\mathbf{V}(14)$	$\mathbf{V}(13)$	$\mathbf{V}(12)$	$\mathbf{V}(123)$	I	$\mathbf{V}(132)$
$\mathbf{V}(13)$	$\mathbf{V}(13)$	$\mathbf{V}(12)$	$\mathbf{V}(14)$	$\mathbf{V}(132)$	$\mathbf{V}(123)$	I

These combination tables show the correspondence (isomorphism) between $\frac{S_4}{\mathbf{V}}$ and S_3.