Proofs of first-order, constant-coefficient linear recurrence relations

First-order, constant-coefficient, linear and homogeneous recurrence relations

These are of the form

$u_{r+1} = au_r$

This is the simplest form of recurrence relation, and its solution is

$u_n = a^n u_0$
where u_0 is the initial value.

Example

A recurrence relation is given by

$u_{r+1} = 3u_r$

If the initial value is 6, find u_7

Solution

$u_n = a^n u_0$

Here $a = 3,\ u_0 = 6,$ hence,

$u_n = 6 \times 3^7 = 13122$

Proof of the formula

To prove:
If $u_{r+1} = au_r$ then $u_n = a^n u_0$
where u_0 is the initial value.

Proof by Mathematical Induction

First step, for $r = 0$

Then LHS = $u_0 = u_0$ = RHS so the first step holds
Induction step

The induction hypothesis is

$$u_k = a^k u_0$$

To prove

$$u_{k+1} = a^{k+1} u_0$$

Then

$$u_{k+1} = a u_k$$

which is the recurrence relation, by substituting for u_k from the induction step, we obtain

$$u_{k+1} = a\left(a^k u_0\right) = a^{k+1} u_0$$

which proves the induction step.

Hence, by mathematical induction the result holds for all n, and

$$u_n = a^n u_0$$

Even more generally, the first-order, linear, constant-coefficient, homogeneous recurrence relation

$$u_{r+1} = a u_r$$

has general solution

$$u_n = B \times a^n$$

where B is a constant

The value of the constant for a particular solution is found by substituting a particular value for which u_n is known.

First-order, constant-coefficient, linear and inhomogeneous recurrence relations

These have general form

$$u_{r+1} = au_r + k$$

and has general solution

$$u_n = Ba^n - \frac{k}{a-1} \qquad \text{if } a \neq 1$$

and

$$u_n = A + nk \qquad \text{if } a = 1$$

Example

Find the general solution to the recurrence relation

$$u_{r+1} = 3u_r - 2$$

and the particular solution if $u_0 = 2$.

Solution

The general solution is

$$u_n = Ba^n - \frac{k}{a-1}$$

where $a = 3$ and $k = -2$

hence,

$$u_n = B3^n - \frac{(-2)}{3-1}$$

Therefore,

$$u_n = B3^n + 1$$

To find the particular solution we substitute, $u_0 = 2,\ n = 0$ to obtain

$$2 = B3^0 + 1$$

$$B = 1.$$

Hence,

$$u_n = 3^n + 1$$

is the particular solution

Proof of the solution to first-order, linear, constant coefficient, inhomogeneous recurrence relations

Let $u_{r+1} = au_r + k$

then,

$u_1 = au_0 + k$

$u_2 = au_1 + k = a\left(au_0 + k\right) + k = a^2u_0 + ak + k$

$u_3 = au_2 + k = a\left(a^2u_0 + ak + k\right) = a^3u_0 + a^2k + ak + k = a^3u_0 + k\left(a^2 + a + 1\right)$

Hence

$u_n = a^n u_0 + k\left(a^{n-1} + a^{n-2} + ... + a + 1\right)$

The expression inside the bracket is the sum of a geometric series

$1,\ a,\ a^2,\ ....,\ a^{n-2},\ a^{n-1}$

with first term 1, ratio a and n terms.

Provided $a \neq 1$ then

$$a^{n-1} + a^{n-2} + ... + a + 1 = \frac{a^n - 1}{a - 1}$$

Hence, if $a \neq 1,$ the solution is

$$u_n = a^n u_0 + k\left(\frac{a^n - 1}{a - 1}\right)$$

Expansion of the bracket gives

$$u_n = a^n u_0 + \frac{ka^n - k}{a - 1}$$

Collecting terms in a^n

$$u_n = a^n\left(u_0 + \frac{k}{a-1}\right) - \frac{k}{a-1}$$

Hence,

$$u_n = Ba^n - \frac{k}{a-1}$$

where B is a constant, as required.

On the other hand, suppose $a = 1,$ then as before

$u_n = a^n u_0 + k\left(a^{n-1} + a^{n-2} + ... + a + 1\right)$

but $a = 1,$ hence

$u_n = 1^n u_0 + k\left(1 + 1 + ... + 1\right)$

where there are n 1s; that is

$u_n = u_0 + nk$

or equivalently,

$u_n = A + nk$

where A is a constant.