Solution of first order differential equations by separation of variables

Prerequisites

You should already be familiar with direct integration and with integration by substitution.

Differential equations

In the relationship $x = f(t)$ we express one variable x as a function of another t. Changes in the variable t bring about changes in x. If we think of x as changing as a consequence of some change in t then x is called the *dependent* variable. The variable that does not change, but rather causes the change, is called the *independent* variable. A differential equation is an equation involving such symbols as

$$\frac{dx}{dt} \qquad \frac{d^2x}{dt^2} \qquad \frac{d^3x}{dt^3} \qquad\qquad f'(t) \qquad f''(t) \qquad f'''(t)$$

and so forth. In a *differential equation* we have an equation in the dependent variable x given in terms of various rates of change of the independent variable t. An example of a differential equation is $3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x + \sin x = 0$. The general aim is to find *exact solutions* to such differential equations. The exact solution is a family of functions of the form $x = f(t) + c$, where c is a constant that satisfies the differential equation. Exact solutions to differential equations cannot always be found. When they cannot be found, the solution may be approximated by a numerical method. Differential equations differ from each other in various systematic ways, and their exact solutions, when they can be found, differ accordingly. A differential equation is called *first-order* if the highest derivative is a first derivative – that is, of the form $\frac{dx}{dt}$.

Example (1)

Which of the following are first order differential equations? State for each the independent and dependent variable.

(a) $\frac{dx}{dt} = 5$

(b) $3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x + \sin x = 0$

(c) $\frac{dP}{dt} = P^2$

(d) $\frac{d^2y}{dx^2} = -kx$

(e) $\frac{ds}{dt} = 4s^2 + s + 1$

Solution

		First order	**independent**	**dependent**
(a)	$\frac{dx}{dt} = 5$	yes	t	x
(b)	$3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x + \sin x = 0$	no	t	x
(c)	$\frac{dP}{dt} = P^2$	yes	t	P
(d)	$\frac{d^2y}{dx^2} = -kx$	no	x	y
(e)	$\frac{ds}{dt} = 4s^2 + s + 1$	yes	t	s

You are already familiar with the solution of a certain type of differential equation.

Example (2)

Solve the equation $\frac{dx}{dt} = 5$.

Solution

This can be solved by direct integration.

$$\frac{dx}{dt} = 5$$

$$x = \int 5\,dt = 5t + c$$

So the most basic method of solving a differential equation is by direct integration. However, this method cannot be used to solve every type of differential equation.

Example (3)

Explain why the first order differential equation $\frac{dP}{dt} = P^2$ cannot be solved by direct integration.

Solution

In $\frac{dP}{dt} = P^2$ the expression P is a function of t. To make this explicit we could write

$P = P(t)$ and $\frac{dP(t)}{dt} = \left[P(t)\right]^2$

Now we see that when we try to integrate with respect to t we get

$P(t) = \int \left[P(t)\right]^2 dt$

As it stands we cannot integrate the right-hand side of this because it involves precisely the function that we are trying to find! Nonetheless, by means of the technique of *separation of variables* the differential equation $\frac{dP}{dt} = P^2$ can be solved.

Solution by separation of variables

For a certain type of first-order differential equation the solution can be found by the technique of the *separation of variables*, which based on integration by substitution. The formula for integration by substitution is given by

Let $x = \int f(t)dt$ and $t = t(u)$ then $\int f(t)dt = \int f(t)\frac{dt}{du}\,du$.

This formula will be very useful when we explain below why the technique of separation of variables works. But integration by substitution, with which you should already be familiar, is best understood by working through examples.

Example (4)

Use integration by substitution with $u = \sqrt{t-1}$ to find $\int t\sqrt{t-1}\,dt$.

Solution

$u = \sqrt{t-1} \quad \Rightarrow \quad u^2 = t - 1 \quad \Rightarrow \quad t = u^2 + 1$

$\frac{dt}{du} = 2u \quad \Rightarrow \quad dt = 2u\,du$

Then

$$\int t\sqrt{t-1}\,dt = \int \left(u^2+1\right)u \times 2u\,du$$

$$= 2\int u^4 + u^2\,du$$

$$= \frac{2}{5}u^5 + \frac{2}{3}u^3 + c$$

$$= \frac{2}{5}\left(\sqrt{t-1}\right)^5 + \frac{2}{3}\left(\sqrt{t-1}\right)^3 + c$$

$$= \frac{2}{15}\left(\sqrt{t-1}\right)(t-1)(3t+2)$$

We also remind you of the proof of the integration by substitution formula.[1]

Example (5)

Prove $x = \int f(t)\,dt$ and $t = t(u)$ then $\int f(t)\,dt = \int f(t)\,\frac{dt}{du}\,du$.

Solution

Let (1) $x = \int f(t)\,dt$ (2) $t = t(u)$

By differentiating both sides of (1) we have $\frac{dx}{dt} = f(t)$.

From (2) $x(t) = x(t(u))$ is a composite function, so applying the chain rule to it

$$\frac{dx}{du} = \frac{dx}{dt} \times \frac{dt}{du}$$

Since $\frac{dx}{dt} = f(t)$

$$\frac{dx}{du} = f(t)\frac{dt}{du}$$

Integrating both sides with respect to u

$$x = \int f(t)\frac{dt}{du}du$$

Substituting for x from (1)

$$\int f(t)\,dt = \int f(t)\frac{dt}{du}du$$

Integration by substitution uses this result in the **forward** sense. Into the integrand

(1) $x = \int f(t)\,dt$

we make the substitution

[1] This proof was optional in the chapter *Integration by substitution*. Hence, example (5) may be omitted

(2) $t = t(u)$

to obtain another integrand

(3) $\int f(t)\frac{dt}{du}du$

that we can directly integrate with respect to *u*. (See example 4.) However, we can **reverse** this process as well. That is the formula $\int f(t)dt = \int f(t)\frac{dt}{du}du$ allows us to cancel out the *du* expressions

$$\int f(t)\frac{dt}{du}du = \int f(t)\frac{dt}{\cancel{du}}\cancel{du} = \int f(t)dt$$

This observation enables us to solve a different type of differential equation.

Example (6)

By direct integration of both sides of

$$\frac{1}{x}\frac{dx}{dt} = t$$

with respect to *t* and use of the formula

$$\int f(t)\frac{dt}{du}du = \int f(t)\frac{dt}{\cancel{du}}\cancel{du} = \int f(t)dt$$

solve $\frac{1}{x}\frac{dx}{dt} = t$.

Solution

$$\frac{1}{x}\frac{dx}{dt} = t$$

Integrate both sides with respect to *t*

$$\int \frac{1}{x}\frac{dx}{dt}\,dt = \int t\ dt$$

Cancel through the expressions *dt* on the left-hand side of this $\int \frac{1}{x}\frac{dx}{\cancel{dt}}\cancel{dt} = \int \frac{1}{x}dx$ to get

$$\int \frac{1}{x}\,dx = \int t\ dt$$

Now integrate both sides, the left-hand side with respect to *x*, the right-hand side with respect to *t*

$$\ln x = \frac{1}{2}t^2 + c$$

Then solve for *x*

$$x = e^{\frac{1}{2}t^2 + c}$$

$$x = e^c e^{\frac{1}{2}t^2}$$

$$x = Ae^{\frac{1}{2}t^2} \qquad A = e^c$$

Note at the last step here c is an arbitrary constant that we introduced at the moment we integrated. $A = e^c$ is another way of writing this constant of integration.

The technique for separation of variables allows us to **step directly** from the line

$$\frac{1}{x}\frac{dx}{dt} = t$$

to the line

$$\int \frac{1}{x}\,dx = \int t\ dt$$

That is, we nearly always **omit** the two lines

$$\int \frac{1}{x}\frac{dx}{dt}\,dt = \int t\ dt$$ Integrate both sides with respect to t

$$\int \frac{1}{x}\frac{dx}{\not{dt}}\,\not{dt} = \int t\ dt$$ Cancel through dt on the left-hand side to obtain an integral in dx

This technique enables us to solve a large new class of differential equations. It can be generalised as follows.

Technique of separation of variables

Suppose we have

$$\frac{dx}{dt} = k(t) \times h(x)$$

where $k(t)$ and $h(x)$ are functions in t and x respectively. This is a first order differential equation and is solved by bringing all expressions in dx and x to one side of the equation, and all expressions in dt and t to the other side. To do so, divide both sides by $h(x)$ to get

$$\frac{1}{h(x)} \times \frac{dx}{dt} = k(t) \qquad (1)$$

Then integrate both sides with respect to t.

$$\int \frac{1}{h(x)}\,dx = \int k(t)\ dt \qquad (2)$$

Both sides can now be separately integrated – the left-hand side as a function of x and the right-hand side as a function of t.

Remark

In the above we omitted the following lines that justify the technique

$$\frac{1}{h(x)}\frac{dx}{dt} = k(t) \qquad (1)$$

$$\int \frac{1}{h(x)}\frac{dx}{dt}\,dt = \int k(t)\,dt$$

$$\int \frac{1}{h(x)}\frac{dx}{\cancel{dt}}\,\cancel{dt} = \int k(t)\,dt$$

$$\int \frac{1}{h(x)}\,dx = \int k(t)\;dt \qquad (2)$$

In practice it is usual to step from line (1) to line (2) without the intermediate steps.

Example (7)

By means of separation of variables solve $\frac{dP}{dt} = P^2$.

Solution

$$\frac{dP}{dt} = P^2 \qquad (1)$$

$$\frac{1}{P^2}\,\frac{dP}{dt} = 1$$

$$\int \frac{1}{P^2}\,dP = \int dt \qquad (2)$$

$$-P^{-1} = t + c \qquad c = \text{constant}$$

$$\frac{1}{P} = c - t$$

$$P = \frac{1}{c - t}$$

Remark

In this solution it is also usual to step directly from the line at (1) to the line at (2) and omit the intervening line. Of course, it is also perfectly permissible to include the intervening line.

Example (8)

Solve $\frac{dy}{dx} = \frac{x}{y^2 - 2}$.

Solution

$$\frac{dy}{dx}=\frac{x}{y^2-2}$$

$$\int\left(y^2-2\right)dy=\int x\,dx$$

$\dfrac{y^3}{3}-2y=\dfrac{x^2}{2}+c$ where c is a constant

$2y^3-12y=3x^2+k$ where $k=6c$ is a constant

Remark

There are two points about this solution to note

(1) The constant of integration is arbitrary; therefore, expressions like *6c* are not necessary. We keep only one constant of integration at any one point.

(2) Exact solutions of the form $y=f(x)$ are not always possible. The final solution can be expressed in terms of any relationship between y and x that is reasonably "neat".

The next example illustrates the process of finding particular solutions to differential equations. This means, to find the exact value for the constant(s) in the differential equation. In order to find a particular solution we need more information. As the general solution to a differential equation involves one or more constants, different values for these constants define different particular solutions. However, all these solutions are related as they are particular examples of the general solution. The general solution, therefore, represents a family of functions. Each member of the family has a different but related graph. Each graph passes through various points, so information about which points a graph passes through enables one to find the particular solution. The additional pieces of information are known as the *initial conditions* or *boundary conditions*.

Example (7)

The gradient function of a curve $\dfrac{dy}{dx}$ is proportional to $(2x+4)$. If the curve passes through the origin and the point $(2,4)$ find its equation.

Solution

$\dfrac{dy}{dx}\propto(2x+4)$ $\dfrac{dy}{dx}$ is proportional to $(2x+4)$

$\dfrac{dy}{dx}=k(2x+4)$ where $k=$ constant of proportionality

$$\int dy=\int k(2x+4)\,dx$$

$y=k\left(x^2+4x\right)+c$ $c=$ constant of integration

We have two constants, the original constant of proportionality and the constant introduced through integration. To find the particular solution and determine the values of these constants we substitute the values from the initial conditions.

(1) Since the curve passes through the origin we have $x = 0 \quad y = 0$. Substituting these values into

$y = k\left(x^2 + 4x\right) + c \qquad\qquad k,c =$ constants

gives $c = 0$.

(2) We are also given $x = 2$ when $y = 4$. Substituting these values gives

$4 = k\left(4 + 8\right) + c$

$k = \frac{4}{12} = \frac{1}{3}$

The particular solution is

$y = \frac{1}{3}\left(x^2 + 4x\right)$

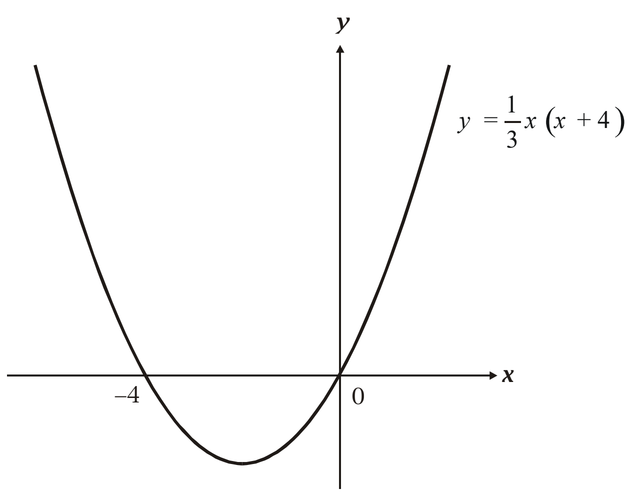
The general solution to this differential equation is

$y = kx\left(x + 4\right) + c \qquad\qquad k,c =$ constants

The particular solution is

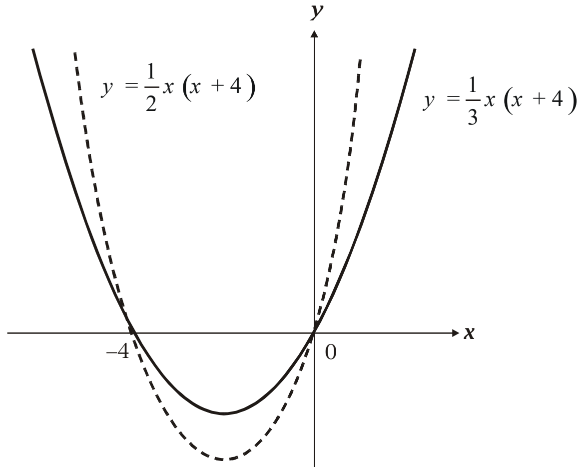
$y = \frac{1}{3}x\left(x + 4\right)$

The graph of $y = \frac{1}{3}x\left(x + 4\right)$ is shown below.



Had the initial conditions been different, the final particular solution would also have been different. But examination of the form of the general solution $y = kx\left(x + 4\right) + c$ shows that all the particular solutions would be related in some way.

For example, as the constant k increases, the shape of the parabola gets steeper, but it still crosses the x-axis at –4 and 0.



On the other hand, if c increases then the whole parabola shifts upwards. So general solutions to differential equations are a family of related functions, represented by a series of curves with similar properties. The particular solution is just one of this family of curves.

Finally we will illustrate a solution to a differential equation where the equation involves a trigonometric function

Example (9)

Solve $x\frac{dx}{dy} = 2\cos^2 y$

Solution

$$x\frac{dx}{dy} = 2\cos^2 y$$

$$\int x\,dx = \int 2\cos^2 y\,dy$$

To find $\int 2\cos^2 y\,dy$

$\cos 2y = \cos^2 y - \sin^2 y = 2\cos^2 y - 1$ Double angle formula

$\cos^2 y = \frac{1}{2}\cos 2y + \frac{1}{2}$ Rearrangement

$$\int 2\cos^2 y\,dy = \int (\cos 2y + 1)\,dy = \frac{1}{2}\sin 2y + y + c$$

$$\frac{x^2}{2} = \frac{1}{2}\sin 2y + y + c$$

$x^2 = \sin 2y + 2y + k$ $k = 2c$