First Order Linear Differential Equations Involving an Integrating Factor

A differential equation is an equation involving the derivatives of functions – that is, expressions of the form:

$$\frac{dx}{dt}, \quad \frac{d^2x}{dt^2}$$

A differential equation is called first-order if the highest derivative is a first derivative – that is, of the form:

$$\frac{dx}{dt}$$

For a certain class of first-order differential equations, the function that gives rise to the equation can be found by the technique of the separation of variables. That technique is described in another section of this course.

In this section we investigate techniques for the solution of first-order differential equations where the variables are not separable. The general form of such an equation is:

$$\frac{dy}{dx} + l(x) \times y = k(x)$$

To be first order the highest power of y must also be first-order – that is, expressions of the form y^2 are not dealt with here. The expressions $l(x)$, $k(x)$ denote functions of x that do not involve y. The expression is linear because it involves the sum of expressionseach of which is linear. The variables cannot be separated because of the presence of the term, $l(x) \times y$.

An example of a first-order linear differential equation is:

$$\frac{dy}{dx} + 3y = e^{2x}$$

Here, $l(x) = 3$ and $k(x) = e^{2x}$. We will integrate this expression later on.

Integration of these equations is achieved by reversing the process of differentiation when that differentiation is by means of the Product rule. Recall that differentiation by means of the Product Rule is given by:

$$(f \times g)' = f' \times g + f \times g'$$
$$\frac{d(uv)}{dx} = \frac{du}{dx} \times v + \frac{dv}{dx} \times u$$

For differentiation, the Product Rule is most conveniently expressed in function notation – that is, in the notation of the first of these expressions. However, when reversing the process in the technique of the integrating factor for the solution of first-order differential equations, which we are studying here, we use the Leibniz notation. That is, the second form of notation.

$$\frac{d(uv)}{dx} = \frac{du}{dx} \times v + \frac{dv}{dx} \times u$$

Thus, supposing we have a differential equation of the form:

$$\frac{du}{dx} \times v + \frac{dv}{dx} \times u = k(x)$$

Then we can integrate directly to

$$u \times v = \int k(x)\ dt$$

The expression $\frac{du}{dx} \times v + \frac{dv}{dx} \times u = k(x)$ is called an exact first-order differential equation.

It is exact because it can be directly integrated as the reverse process of differentiation by the Product Rule.

Example

$$2xy\frac{dy}{dx} + y^2 = e^{2x}$$

integrates directly to

$$xy^2 = \int e^{2x}\ dx$$

$$\therefore xy^2 = \frac{1}{2}e^{2x} + A$$

Here $u(x) = x,\ v(x) = y^2$

$$\frac{du}{dx} = 1,\ \ \frac{dv}{dx} = 2y\frac{dy}{dx}$$

Integrating Factor

Sometimes a first-order differential equation is not exact, but it is possible to "spot" a function that, if used to multiply every term, will give an exact differential equation. This function is called an integrating factor.

For example,

$$x\frac{dy}{dx} + 2y = e^{x^2}$$

This is not exact, but multiplying every term by the integrating factor, *x*, gives:

$$x^2\frac{dy}{dx} + 2xy = xe^{x^2}$$

Then, $x^2 y = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + c$

Another example is:

$$\left(x + y^2\right)\frac{dy}{dx} - y = 0$$

The integrating factor is $1/y^2$

Then, $\frac{x + y^2}{y^2} \times \frac{dy}{dx} - \frac{1}{y} = 0$

Hence, $\frac{x}{y} - y = 0$

Recognising integrating factors directly can be tricky! However, we shall introduce a technique that finds the integrating factor without the need for inspired intuition! But the technique of direct integration as the inverse of the Product Rule is the fundamental concept here, and awareness of this can save time, and be particularly important in other applications.

Example

Find the general solution of the differential equation:

$$\frac{dy}{dx} + y\cot x = x.$$

It must first be appreciated that $\cot x$ can be written as ${\cos x}/{\sin x}$, and then that $\sin x$ is the integrating factor:

$$\frac{dy}{dx} + y\frac{\cos x}{\sin x} = x$$

$$\therefore \sin x\frac{dy}{dx} + y\cos x = x\sin x$$

$$\therefore y\sin x = \int x\sin x \; dx$$

The integral of $x\sin x$ is found by integrating by parts. The parts formula is:

$$\int fg' = fg - \int f'g$$

So, let:

$$f(x) = x, \quad g'(x) = \sin x$$

then,

$$f'(x) = 1, \quad g(x) = -\cos x$$

$$\therefore \int x\sin x \; dx = -x\cos x - \int -\cos x \; dx = -x\cos x + \sin x + c$$

Hence, $\quad y\sin x = -x\cos x + \sin x + c$

General technique for integrating first-order differential equations

We now describe the general technique for integrating first-order differential equations. Using this technique we would be able to integrate the above example without the use of direct intuition.

If a first-order differential equation can be put into the form

$$\frac{dy}{dx} + l(x)y = k(x)$$

Then the integrating factor is $p(x) = e^{\int l(x).dx}$

Then, multiplying the differential equation by the integrating factor gives:

$$e^{\int l(x).dx} \times \frac{dy}{dx} + l(x) \times y \times e^{\int l(x).dx} = k(x) \times e^{\int l(x).dx}$$

Then direct integration gives:

$$ye^{\int l(x).dx} = \int k(x).e^{\int l(x).dx} dx$$

We can show that $p(x) = e^{\int l(x).dx}$ is the integrating factor since,

$$\frac{d}{dx}\left(ye^{\int l(x).dx} \right) = \frac{dy}{dx} e^{\int l(x).dx} + ye^{\int l(x).dx} \times l(x)$$

<u>Example</u>

An example of the use of this formula is:

Solve: $\frac{dy}{dx} + 3y = e^{2x}$

Here, $l(x) = 3$,

$\therefore p(x) = e^{\int 3.dx} = e^{3x}$

Then $e^{3x}\frac{dy}{dx} + 3ye^{3x} = e^{2x} \times e^{3x}$

$\therefore ye^{3x} = \int e^{5x} dx$

$\therefore ye^{3x} = \frac{1}{5}e^{5x} + c$

In the case of $\frac{dy}{dx}+y\cot x=x$

The integrating factor is $p(x)=e^{\int \cot x.dx}$

Writing $\cot x$ as $\frac{\cos x}{\sin x}$

$\int \cot x=\ln(\sin x)$

Hence, $p(x)=e^{\ln(\sin x)}=\sin x$.

We solved this problem earlier in this section, and having found the integrating factor, the rest of the solution is identical to the last.

Initial Conditions

The solution to a first-order differential equation introduces a constant of integration. Consequently, the solution is a one-dimensional family of functions differing from each other by this constant of integration.

Hence, substituting particular values of the dependent and independent variable solves a particular problem in physics that gives rise to a differential equation of this type.

Example

Find the particular solution of

$$y^2\frac{dy}{dx}=x^2+1$$

given that $y=1$ when $x=2$.

$$\int y^2dy=\int x^2+1.dx$$

$$\frac{y^3}{3}=\frac{x^3}{3}+x+c$$

For the particular solution, we substitute $y=1,\ x=2$.

Then

$$\frac{1}{3} = \frac{8}{3} + 2 + c$$

$$c = -\frac{7}{3} - \frac{6}{3} = -\frac{13}{3}$$

Therefore,

$$\frac{y^3}{3} = \frac{x^3}{3} + x - 4\frac{1}{3}$$