Formation of a differential equation

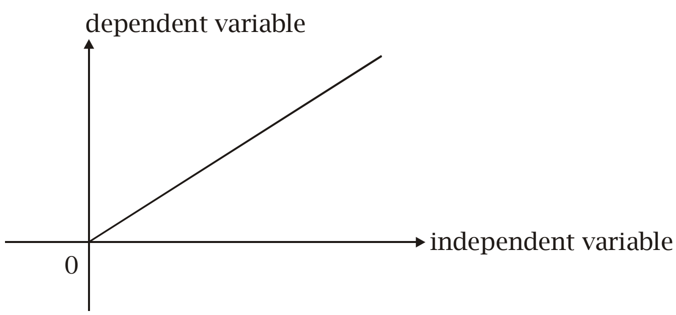
Prerequisites

You should be familiar with direct proportionality and be able to integrate a first-order differential equation by the technique of separation of variables.

Proportionality

A variable represents a physical quantity that can take more than one value. For example, time and distance are variables. Proportionality is concerned with particular relationships between variables. As one variable changes, so too does the other. When we consider proportionality the way one variable changes with another can be represented in a graph as a straight line. We call such relationships *linear*. Proportionality is a linear relationship between two variables.

Generally, in these relationships we think of one variable as changing as a consequence of some change in the other variable. The variable that changes as a consequence of changes in the other variable is called the *dependent* variable. The variable that does not change, but rather causes the change, is called the *independent* variable. For example, the acceleration of a car (dependent variable) is related to the pressure on the throttle (independent variable). In *direct proportionality*, an increase in the independent variable causes a linear increase in the dependent variable. This is represented graphically as a straight-line relationship with the independent variable along the horizontal axis and the dependent variable along the vertical axis.



The graph is a straight line through the origin. It is clear that the line representing this relationship has a constant gradient, k. The symbol for proportionality is $\propto$. The expression,

$y \propto x$

means, "y is proportional to x". The gradient is constant, hence

If $y \propto x$ then $y = kx$ where $k = \text{constant}$.

Example (1)

In a kitchen time taken (*t*) prepare the food for a party is proportional to the number of guests (*n*). If it takes 4 hours to prepare for 100 guests, express the relationship between the time taken and the number of guests as an equation, and find the time that it will take to prepare for 1025 guests.

Solution

$t \propto n$

$t = kn$

$k = \frac{t}{n}$

$n = 100,\ t = 4 \quad \Rightarrow \quad k = \frac{4}{100} = 0.04$

$t = 0.04n$

$n = 1025 \Rightarrow \quad t = 0.04 \times 1025 = 41$ hours

So far we have considered the possibility where *y* varies directly as *x*, but there is the possibility that *y* will vary as some power of *x* varies. For example, *y* could be proportional to the square of *x*. We represent this as $y \propto x^2$. And in general if $y \propto x^n$ then $y = k\,x^n$ where k = constant. Relationships of proportionality can also be combined. Thus If $y \propto x^n$ and $y \propto z^m$ then $y \propto x^n z^m$ and $y = k\,x^n z^m$ where *k* is a constant.

Differential equations

A differential equation is any equation involving a derivative. The expression

$$\frac{dy}{dx} = e^{-2x}$$

is a differential equation. We solve this by integrating both sides.

$$y = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + c$$

In other words, whenever we integrate we are in fact solving a differential equation and direct integration is the first technique to be learnt for solving differential equations.

Example (2)

Solve the differential equation

$$\frac{dy}{dx} = \sin 2x + \frac{1}{x}$$

Solution

By direct integration

$$y = -\frac{1}{2}\cos 2x + \ln x + c$$

A first order differential equation involves only first order derivatives.

$\frac{dy}{dx} = e^{-2x}$ is first order, but $\frac{d^2y}{dx^2} = e^{-2x}$ is not.

In addition to direct integration, you should already have met the technique of separation of variables as a means of solving first order differential equations.

Example (3)

Solve by separation of variables the first order differential equation

$$\frac{dx}{dt} = x(t-2)$$

Solution

$$\frac{dx}{dt} = x(t-2)$$

$$\int \frac{1}{x}\,dx = \int (t-2)\,dt$$

$$\ln x = \frac{1}{2}t^2 - 2t + c$$

$$x = e^{\frac{1}{2}t^2 - 2t + c}$$

$$x = e^c e^{\frac{1}{2}t^2 - 2t}$$

$$x = Ae^{\frac{1}{2}t^2 - 2t} \qquad A = e^c$$

Models

Many physical laws involve the notion of a *rate of change.* Newton's famous second law expresses the way in which physical forces cause objects to accelerate.

$F = ma$ Force = mass $\times$ acceleration.

Since acceleration is defined to be the rate of change of velocity with respect to time, then Newton's second law is a statement about rates. We could rewrite it as, *The rate of change of velocity of a particle is proportional to the force applied to it.* Using the symbol $\propto$ to stand for proportionality and $\frac{dv}{dt}$ to stand for the rate of change of velocity, *v*, with respect to time, *t*., this becomes $\frac{dv}{dt} \propto F$. When one physical quantity is proportional to another we can introduce a *constant of proportionality*. In the case of Newton's second law this constant is mass, *m*. His law becomes

$$F = m\frac{dv}{dt}$$

This shows that Newton's second law is a statement about the rate of change of a physical quantity, here velocity, and leads to a differential equation. We use the term *model* to apply to any situation in which we represent the relationship between physical quantities from the real world by a mathematical structure of any kind. A differential equation is a particular type of mathematical structure. Thus Newton's second law is a *model* of the physical motion of objects. This model is so useful and so accurate that it was used to plan the Apollo mission to land a man on the Moon.

Forming and solving first order differential equations

The expression, "The rate of change of the dependent variable *x* with respect to the independent variable *t* is constant" is translated as

$$\frac{dx}{dt} = k$$

Example (4)

Speed is the rate of change of distance, *x*, with respect to time, *t*. Translate the expression, "He ran at a constant speed of 5 ms^{-1}" into mathematical symbols.

Solution

$$\frac{dx}{dt} = 5 \text{ ms}^{-1}$$

The expression, "The rate of change of a dependent variable, say *x*, with respect to another variable, say *t*, is proportional to a function of the independent variable $f(t)$" is translated as

$$\frac{dx}{dt} \propto f(t)$$
$$\frac{dx}{dt} = kf(t) \qquad\qquad k = \text{constant}$$

Example (5)

Newton's exponential law of cooling states that the rate of change of the temperature of a cooling body is proportional to the difference between the temperature of the body and the temperature of its surroundings. Let θ be the difference between the temperature of the cooling body and its surroundings and let t be time. Formulate a differential equation.

Solution

$$\frac{d\theta}{dt} \propto -\theta$$

We introduce the negative sign because the water is cooling. Hence

$$\frac{d\theta}{dt} = -k\theta$$

where $k > 0$ is the constant of proportionality.

Remark

Whenever you translate a statement in words about the relationship between a dependent and independent variable into mathematical symbols you are making a *model* of that physical relationship.

In these examples we have formed differential equations. As only first order derivatives, $\frac{dx}{dt}$, are involved in them, the differential equations formed are examples of *first-order differential equations.*

Example (5) continued

Newton's exponential law of cooling states that the rate of change of the temperature, θ, of a cooling body is proportional to the difference between the temperature of the body and the temperature of its surroundings.

(*a*) Formulate a differential equation.

(*b*) A bucket of hot water is cooling. Its temperature at 0 minutes is 90°C and at 8 minutes it is 70°C. Find the temperature of the water at 20 minutes. The ambient temperature (temperature of the surroundings) is 25°C.

Solution

(a) As before

$$\frac{d\theta}{dt} = -k\theta$$

where θ is the difference in the temperature of the body and the temperature of the surroundings.

(b) This is solved by separation of variables

$$\int \frac{d\theta}{\theta} = -\int k\,dt$$

$$\ln\theta = -kt + c$$

In order to find the constants, we substitute the initial conditions

When $t = 0$, $\theta = 90 - 25 = 75$, hence, $c = \ln 75$

When $t = 8$, $\theta = 70 - 25 = 45$; therefore

$$\ln 45 = -8k + \ln 75$$

$$k = \frac{\ln 75 - \ln 45}{8} = 0.0638...$$

When $t = 20$

$$\ln\theta = -0.0638 \times (20) + \ln 75 = 3.0404...$$

$$\theta = 20.9°C \quad (\text{nearest } 0.1°)$$

Temperature of the body $20.9 + 25 = 45.9°C \quad (\text{nearest } 0.1°)$

Example (6)

The size P of the population of a habitation is modelled as a continuous function of the real variable t, where t is in years. The rate of increase of P is directly proportional to P.

(a) Write down a differential equation that is satisfied by P.

(b) Given that the initial size of the population is P_0, show that $P = P_0 e^{kt}$, where k is a positive constant.

(c) The initial size of the population at the foundation of the habitation is 15. At 120 years the population is $P_1 = 75$. Determine P_0 and k in $P = P_0 e^{kt}$. Give your answer for k as an exact logarithm.

(d) The habitation may be defined to be a metropolis when the population reaches 1,000,000 inhabitants. In what year after the foundation will the habitation become a metropolis?

Solution

(a) $\frac{dP}{dt} \propto P \quad \Rightarrow \quad \frac{dP}{dt} = kP \qquad\qquad k = \text{constant}$

(*b*) $$\frac{dP}{dt} = kP$$
$$\int \frac{1}{P} dP = \int k dt$$
$$\ln P = kt + c$$
$$P = e^{kt+c} = e^c e^{kt}$$
When $t = 0,\ P = P_0 \Rightarrow e^c = P_0$. Hence
$$P = P_0 e^{kt}$$

(*c*) We are given $P_0 = 15$ therefore we have $P = 15e^{kt}$

When $t = 120$ we have $P = P_1 = 75$; therefore, on substituting into $P = 15e^{kt}$

$$75 = 15e^{120k}$$
$$e^{120k} = 5$$
$$120k = \ln(5)$$
$$k = \frac{1}{120}\ln(5)$$

(*d*) $$10^6 = 15e^{\left(\frac{1}{120}\ln(5)\right)t}$$
$$\frac{10^6}{15} = e^{\left(\frac{1}{120}\ln(5)\right)t}$$
$$\ln\left(\frac{10^6}{15}\right) = \left(\frac{1}{120}\ln(5)\right)t$$
$$t = 120 \times \frac{\ln\left(\frac{10^6}{15}\right)}{\ln(5)} = 828.17\ldots$$

The answer is the next whole integer up, $t = 829$ years. That is, in the 829th year after the foundation.