Functions

Prerequisites

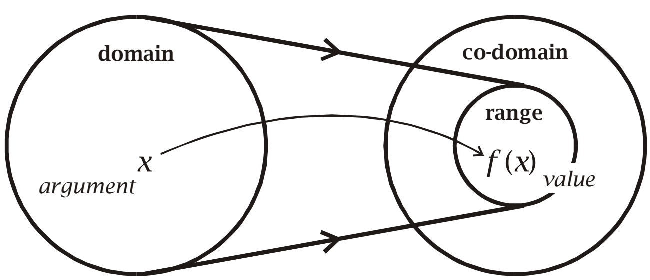
You should be familiar with the idea of a *function* as a *mapping* from one set (called the *domain*) to another set (called the *co-domain*). That is, a function is a rule taking you from one number to another. For a given application of a rule, the number in the domain is called the *argument* of the function and the number to which it is mapped by the rule is called its *value*. In the example

$$f\begin{cases}\mathbb{R} \to \mathbb{R}^+ \\ x \to e^x\end{cases}$$

the first and second lines are interpreted as follows

$$f\begin{cases}\text{domain } \textit{maps to} \text{ co-domain} \\ \text{argument } \textit{maps to} \text{ value}\end{cases}$$

So in this case the domain is the set $\mathbb{R}$ standing for all real numbers, the co-domain is the set $\mathbb{R}^+$ standing for all positive real numbers, which is the same as the interval $x > 0$. The argument is the variable x and the value is e^x. The expressions $f : x \to e^x$ and $f(x) = e^x$ are equivalent – they mean the same thing so that each can replace the other. The *image* or *range* of a function $y = f(x)$ is the set of all numbers in the co-domain that can be values of the function. In set notation range $= \{y : y = f(x) \text{ and } x \in \mathbb{R}\}$, which is read "the range of f is the set of all numbers y such that y is a value of $f(x)$ for some argument x in the domain of f". The terms *image* and *range* are equivalent; they mean the same thing and "image" can be used instead of "range" and vice-versa. There is no general requirement that the co-domain should be equal to the range. The range must be contained in the co-domain, but the co-domain may be larger than the range. However, the domain cannot contain numbers x for which the function $f(x)$ is undefined.



Example (1)

A function is defined by

$$f\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\to\cos x+1\end{cases}$$

(*a*) State the domain and co-domain of f

(*b*) What is the value of $f(x)$ when its argument is $60°$?

(*c*) Sketch the graph $y=f(x)$

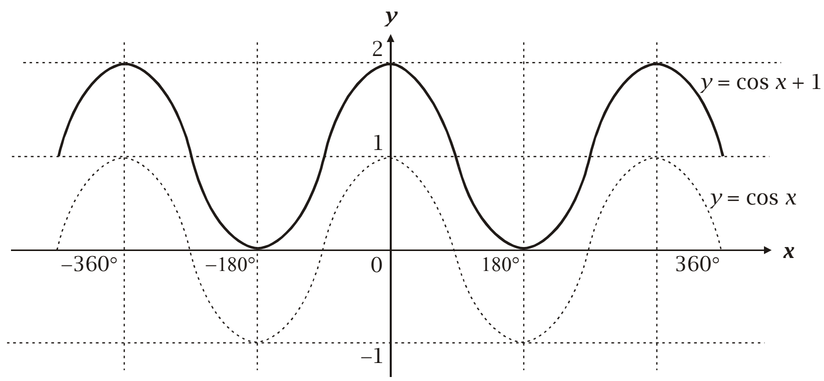
(*d*) Write down the range of *f*.

Solution

(*a*) Domain = co-domain = $\mathbb{R}$

(*b*) $f(60°)=\cos(60°)+1=0.5+1=1.5$

(*c*) The graph of $y=\cos x+1$ is the graph of $y=\cos x$ shifted vertically up 1.



(*d*) Range = $0\le x\le 2$ or the closed interval $[0,2]$.

The expressions like $[0,\infty)$ and $(-\infty,\infty)$ which denote intervals use the convention that a square (closed) bracket means that the point next to it is *included* in the set, and a curved (open) bracket means that the point next to it is *not included*. The symbol ∞ is used to denote infinity; as infinity is not a number then it cannot be included in the set, so we use a curved (open) bracket next to it.

Composition of Functions

To compose functions means to follow one function by another – to apply one function to the result of another.

Example (2)

Let f and g be the functions

$$f\begin{cases}\mathbb{R} \to \mathbb{R} \\ x \to x+2\end{cases} \qquad g\begin{cases}\mathbb{R} \to \mathbb{R} \\ x \to 3x\end{cases}$$

(a) Find

(i) $t = f(5)$

(ii) $g(t)$

(b) Fill in the spaces in the diagram

$2 \xrightarrow{f} \ldots\ldots \xrightarrow{g} \ldots\ldots$

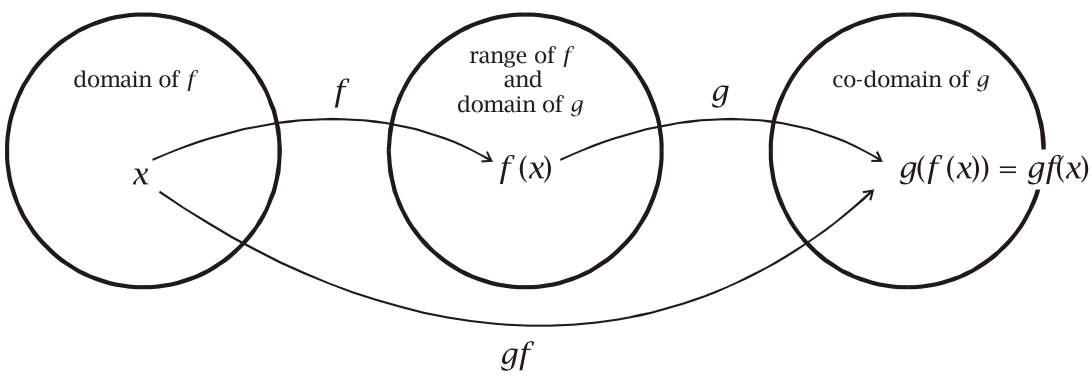
Solution

(a) (i) $t = f(5) = 2+5 = 7$

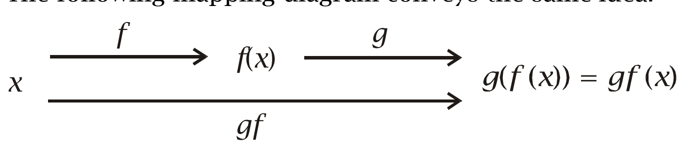
(ii) $g(t) = g(7) = 3 \times 7 = 21$

(b) $2 \xrightarrow{f} 7 \xrightarrow{g} 21$

This example illustrates the composition of functions. We can show the effect of following one function by another by the following diagram.



The following mapping diagram conveys the same idea.



The expression gf stands for the single composite function that is the result of f followed by g. The diagrams above give the meaning of

$gf(x) = g(f(x))$.

The expression $g(f(x))$ is read "g of f of x". To find $g(f(x))$ first evaluate $f(x)$ and then apply g to this value.

Example (3)

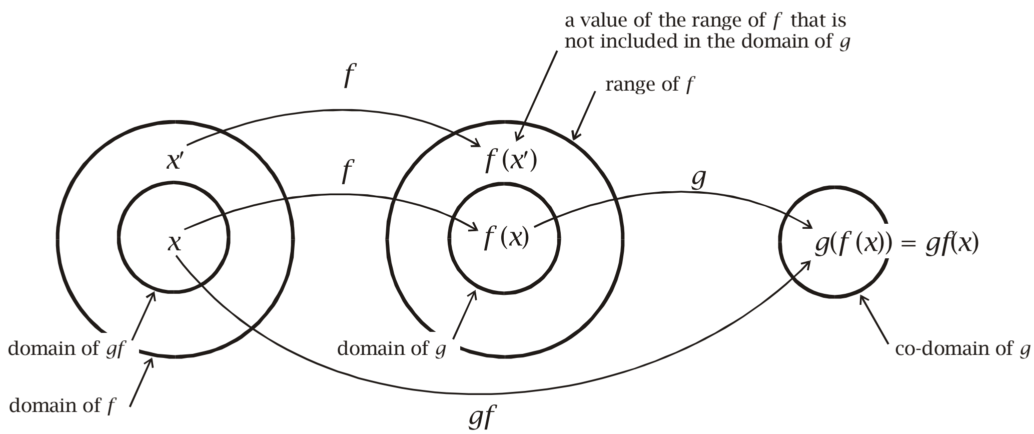
Let $f(x) = 3x + 2$, $g(x) = x^2 + 1$, find $gf(x)$

Solution

$f(x) = 3x + 2 \qquad g(x) = x^2 + 1$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(3x+2) \\ &= (3x+2)^2 + 1 \\ &= 9x^2 + 12x + 4 + 1 \\ &= 9x^2 + 12x + 5 \end{aligned}$$

Care must be taken over the definition of the domain of a composite function. If the range of *f* is strictly larger than the domain of *g* there will be values *x* in the domain of *f* for which there will be no corresponding value of $gf(x)$. In such a case the domain of *f* must be restricted so as to include only those arguments *x* whose values $f(x)$ are elements of the domain of *g*.



The domain of *gf* must be a set contained in the domain of *f* such that the range of this set is equal to the domain of *g*.

Example (4)

The function *f* has domain $(2, \infty)$ and is defined by $f(x) = \ln(x-2)$

The function *g* has domain $(-\infty, \infty)$ and is defined by $f(x) = e^{-x}$

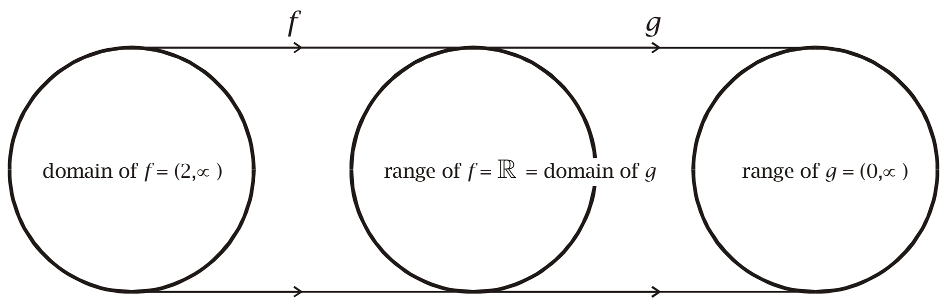
(*a*) State the domain of *gf* and solve the equation $gf(x) = 4$ giving your answer as a fraction.

(*b*) State the domain of *fg* and solve the equation $fg(x) = -4$ giving your answer to 3 significant figures.

Solution

Function	*Mapping*	*Domain*	*Range*
f	$y=\ln(x-2)$	$(2,\infty)$ or $x>2$	$\mathbb{R}=(-\infty,\infty)$
g	$y=e^{-x}$	$\mathbb{R}=(-\infty,\infty)$	$(0,\infty)$ or $x>0$

(*a*) The following diagram illustrates the solution to this part of the question.

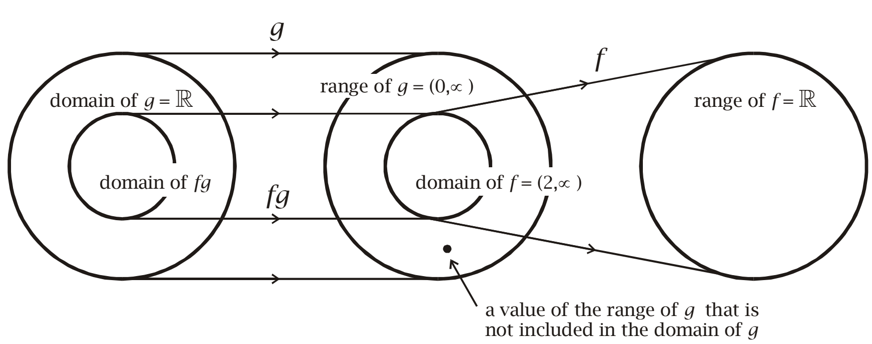


The domain of gf is the same as the domain of f which is $(2,\infty)$ or $x>2$.

For the second part of the question

$$gf(x)=4$$
$$g(f(x))=4$$
$$g(\ln(x-2))=4$$
$$e^{-\ln(x-2)}=4$$
$$\left(e^{\ln(x-2)}\right)^{-1}=4$$
$$(x-2)^{-1}=4$$
$$\frac{1}{x-2}=4$$
$$1=4(x-2)$$
$$1=4x-8$$
$$4x=9$$
$$x=\frac{9}{4}$$

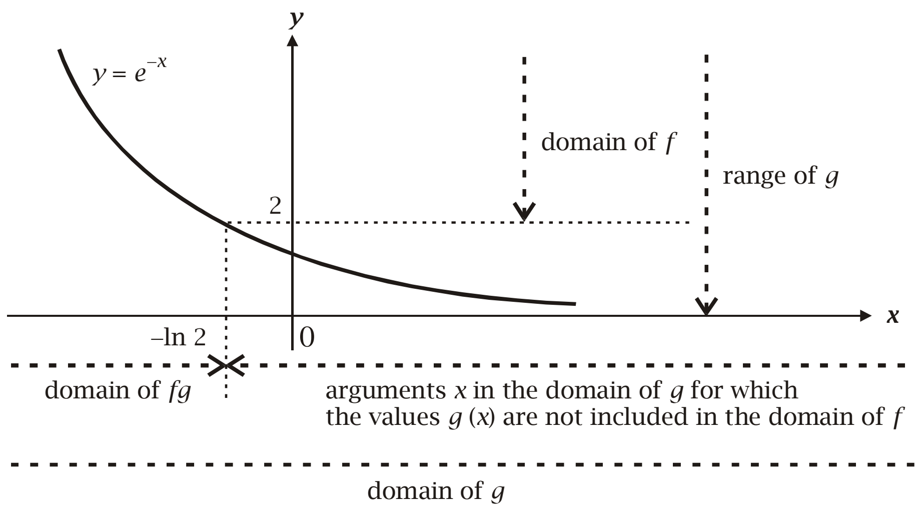
(*b*)



The domain of fg is a subset of domain of g. Since the domain of f is $(2,\infty)$ or $x > 2$ any argument x in the domain of g that maps to a value in the interval $(0,2]$ or $0 < x \le 2$ cannot be included in the domain of gf. We require

$$g(x) > 2$$
$$e^{-x} > 2$$
$$-x > \ln 2$$
$$x < -\ln 2$$
$$x < -0.693 \text{ (3 s.f.)}$$

The following diagram further illustrates this solution.



$$fg(x) = -4$$
$$f\left(e^{-x}\right) = -4$$
$$\ln\left(e^{-x} - 2\right) = -4$$
$$e^{-x} - 2 = e^{-4}$$
$$e^{-x} = e^{-4} + 2$$
$$-x = \ln\left(e^{-4} + 2\right)$$
$$x = -\ln\left(2 + e^{-4}\right) = -0.70226... = -0.702 \text{ (3 s.f.)}$$

Inverse of a Function

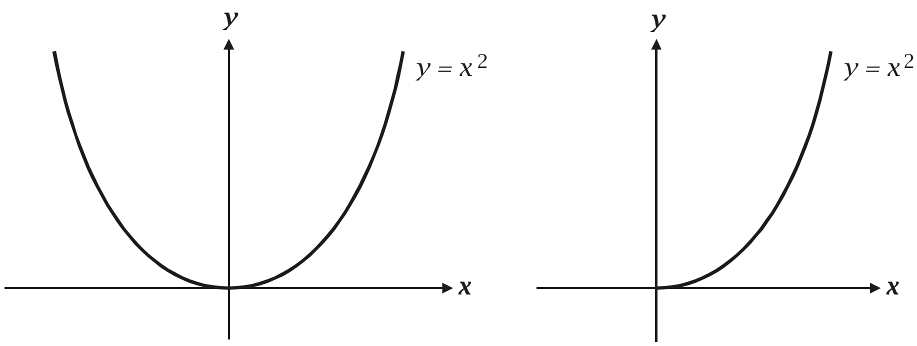
If a function f maps x to y, then the inverse of that function, written f^{-1}, maps y to x. The inverse of a function f reverses the process represented by that function.

$$x = f^{-1}(y) \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y = f(x)$$

You should already be familiar with inverse functions because you should have met many of them. Here are some examples of functions and their inverses that you should already have met.

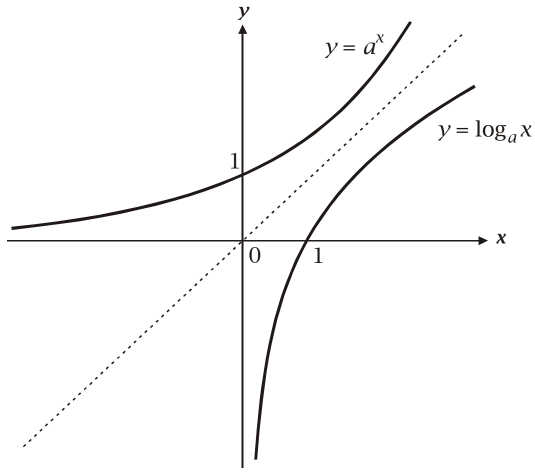
function	inverse
$f(x) = x^2$	$f^{-1}(x) = \sqrt{x}$
$f(x) = \sin x$	$f^{-1}(x) = \sin^{-1} x$
$f(x) = e^x$	$f^{-1}(x) = \ln x$

You should also be aware that for the first two of these examples the inverse can only be obtained by restricting the domain of the function *f*. The domain of $f(x) = x^2$ is the whole interval $\mathbb{R} = (-\infty, \infty)$, but on this interval $f(x) = x^2$ is a many-one function. For a function to have an inverse it must be a one-one function. Thus, in order to define the inverse of $f(x) = x^2$ we must restrict the domain to the interval $\mathbb{R}^+ = [0, \infty)$ or $x \geq 0$.

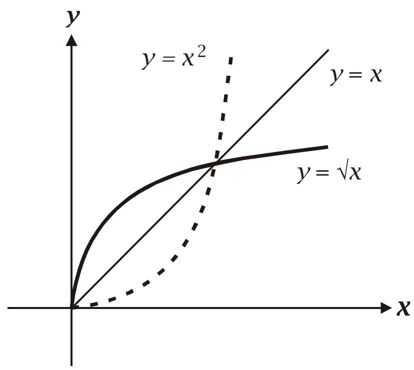


$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to x^2 \end{cases}$ $\xrightarrow[\text{RESTRICTTION}]{}$ $g : \begin{cases} \mathbb{R}^+ \to \mathbb{R}^+ \\ x \to x^2 \end{cases}$

Likewise, for the inverse of $f(x) = \sin x$ the domain has to be restricted to an interval where $f(x) = \sin x$ is a monotone increasing or decreasing function. By convention this is the interval $[180°, 180°]$ or $-180° \leq x \leq 180°$. The domain of $f^{-1}(x) = \sin^{-1} x$ is the interval $[-1, 1]$ or $-1 \leq x \leq 1$ and its range is $[180°, 180°]$ or $-180° \leq x \leq 180°$. The function $f(x) = e^x$ a monotone increasing function and has an inverse $f^{-1}(x) = \ln x$. However, care must be taken when specifying the domain of $f^{-1}(x) = \ln x$ since it is not defined for all real values. The function $f^{-1}(x) = \ln x$ only makes sense for positive values of *x*; hence its domain is the interval $(0, \infty)$ or $x > 0$.



The above diagram demonstrates another fact about inverse functions that you should also already know – that the graph of the inverse function f^{-1} of the function *f* is the reflection in the line $y = x$ of the graph of *f*. The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ in the line $y = x$. Likewise, $y = \sqrt{x}$ is the reflection of the positive part of $y = x^2$ in the line $y = x$.



What this section emphasises is that when defining an inverse care must be taken over the accompanying definition of its domain and co-domain.

Finding the rule for the inverse

If the inverse of a polynomial function exists then it is found by letting $y = f(x)$ and solving for *x*. It is usual to re-express the inverse as a function in terms of *x*.

Example (5)

Given $f(x) = \dfrac{x+3}{3x-7}$, find $f^{-1}(x)$.

Solution

Let $y = f\left(x\right) = \dfrac{x+3}{3x\text{ - }7}$

We must rearrange this equation to make y the subject.

$$y\left(3x-7\right) = x+3$$
$$3xy - 7y = x+3$$
$$3xy - x = 3+7y$$
$$x(3y-1) = 3+7y$$
$$x = \frac{3+7y}{3y-1}$$

Referring to the following diagram

$$x = f^{-1}\left(y\right) \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y = f\left(x\right)$$

we see that we have now found x in terms of y; that is

$$x = f^{-1}(y) = \frac{3+7y}{3y-1}$$

However, in this last example it is important to understand that in the expression

$$f^{-1}(y) = \frac{3+7y}{3y-1}$$

y stands for an arbitrary number in the domain of f^{-1}. Therefore, any letter whatsoever could be used instead of y. For example, t or s.

Example (5) continued

Rewrite $f^{-1}(y) = \dfrac{3+7y}{3y-1}$ using the symbol t instead of y.

Solution

$$f^{-1}(t) = \frac{3+7t}{3t-1}$$

In questions of this type it is usual to demonstrate understanding that the variable letter in the expression is *arbitrary* by rewriting the function in terms of the original variable. In this question we started with x. Therefore the inverse function is written by replacing y by x, to obtain

$$f^{-1}(x) = \frac{3+7x}{3x-1}$$

This is the expected answer in this question, and *not* $f^{-1}(y) = \frac{3+7y}{3y-1}$ even though $f^{-1}(y) = \frac{3+7y}{3y-1}$ is also a correct version of the inverse function.

Example (6)

The function *f* is defined for $x \geq 0$ by $f(x) = 2x^2 - 1$

(*a*) Find an expression for f^{-1}, stating the range and domain of f^{-1}

(*b*) Sketch the graphs of *f* and f^{-1} using the same axes.

Solution

(*a*) Let $y = f(x) = 2x^2 - 1$

$$2x^2 = y + 1$$
$$x^2 = \frac{y+1}{2}$$
$$x = \sqrt{\frac{y+1}{2}}$$
$$f^{-1}(x) = \sqrt{\frac{x+1}{2}}$$

The function $f(x) = 2x^2 - 1$ is defined only for $x \geq 0$ and its range is therefore $[-1,\infty)$ or $x \geq -1$. The function f^{-1} is only defined for positive $\frac{x+1}{2} > 0$, which is also the interval $[-1,\infty)$. Therefore, the domain of f^{-1} is the range of *f*, which is $[-1,\infty)$ or $x \geq -1$. The range of f^{-1} is the domain of *f*, which is $[0,\infty)$ or $x \geq 0$.

(*b*)