Functions and continuity

Functions

A *function* is a *mapping* from one set (called the *domain*) to another set (called the *co-domain*). That is, a function is a rule taking you from one number to another. Functions can be written explicitly by specifying exactly which numbers are mapped to which. For example

$$f:\begin{cases} 0 \to 3 \\ 1 \to 7 \\ 2 \to 10 \end{cases}$$

The symbol $0 \to 3$ is read "0 maps to 3". It is only possible to specify finite functions by such an explicit rule, or mapping diagram. Often we define a function by an implicit rule indicating the process that takes you from a number in the domain to a number in the co-domain.

$$f:\begin{pmatrix} \mathbb{R} \to \mathbb{R} \\ x \to 3x^2+2 \end{pmatrix}$$

This is read "f is the function from $\mathbb{R}$ to $\mathbb{R}$ such that x maps to $3x^2+2$." The symbols $x \to 3x^2+2$ and $f(x)=3x^2+2$ are interchangeable. For a given application of a rule, the number in the domain is called the *argument* of the function and the number to which it is mapped by the rule is called its *value*. For example

$$f(x)=3x^2+2$$
$$f(1)=5$$

Here 1 is the argument and 5 is the value. The image set is the set of all numbers in the co-domain that can be values of the function. In set notation

$$\text{Image } = \{y : y = f(x)\}$$

The image may be equal to or smaller than the codomain. For example

$f(x)=\begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to x^2 \end{cases}$ Domain = $\mathbb{R}$ Codomain = $\mathbb{R}$ Image = $\{x : x \in \mathbb{R} \text{ and } x \geq 0\}$

We could have written the image as $[0, \infty)$. This uses the convention that a square (closed) bracket means that the point next to it is *included* in the set, and a curve bracket means that the point next to it is *not included*. The symbol ∞ is used to denote infinity; as infinity is not a number then it cannot be included in the set, so we use a curved (open) bracket next to it.

Graph

The *graph* of a function *f* from $\mathbb{R}$ to $\mathbb{R}$ is the subset of $\mathbb{R}^2$

$\{(x,y) \in \mathbb{R}^2 : y = f(x)\}$

So it is a set of points in the *x,y*-plane. To a point of *x* there corresponds a point of *y*. This is a formal definition of a graph. By this stage the student should be very familiar informally with graphs and should be used to sketching the graphs of functions.

Example (1)

Constant functions map every point of the domain to a fixed point in the codomain. Let $f(x) = 3$

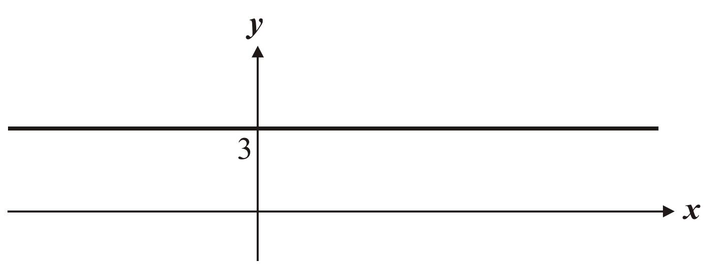
(a) Describe as a set the graph of *f*.

(b) Sketch the graph of *f*.

Solution

(a) The graph is the set $\{(x,3) : x \in \mathbb{R}\}$

(b)



Example (2)

Affine functions are functions that take the form

$$f \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto ax + b \end{cases}$$

where *a, b* are fixed real numbers and $a \neq 0$. Their graphs are straight lines. Let $f(x) = 3x + 1$

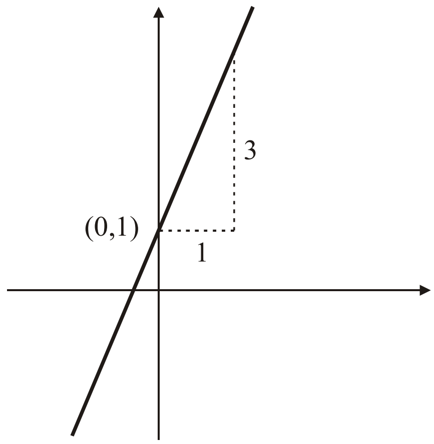
(a) Describe as a set the graph of *f*.

(b) Sketch the graph of *f*.

Solution

(a) The graph is the set $\{(x, 3x+1) : x \in \mathbb{R}\}$

(b) It is the straight line through the point $(0,1)$ with gradient 3.



Example (3)

Quadratic functions take the form

$$f\begin{cases}\mathbb{R} \to \mathbb{R} \\ x \mapsto ax^2 + bx + c\end{cases}$$

where *a, b, c* are fixed real numbers and $a \neq 0$. Let $f(x) = x^2 + 2x - 3$

(a) Factorise *f* and put *f* in completed square form.

(b) Sketch the graph of *f*.

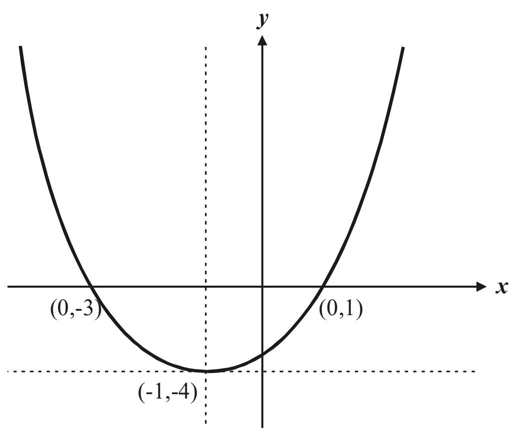
Solution

(a) $f(x) = x^2 + 2x - 3 = (x-1)(x+3)$

Completing the square

$f(x) = x^2 + 2x - 3 = x^2 + 2x + 1^2 - 1^2 - 3 = (x+1)^2 - 4$

(b) This is a parabola with axis of symmetry $x = -1$, minimum point $(-1,-4)$ and cutting the *x*-axis at $y = -3$ and $y = 1$ so its sketch is



Analytic methods of sketching graphs

Polynomial functions take the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n, a_{n-1}, \ldots a_1, a_0$ are fixed real numbers.. You should be familiar by this stage with analytic methods of sketching polynomial functions – that is, by using the differential calculus to find their turning points and to classify these as maxima or minima.

Example (4)

Use analytical methods to sketch the graph of $f(x) = 3 - 4x + 6x^2 - x^3$.

Solution

$y = 3 - 4x + 6x^2 - x^3$

For turning points $\frac{dy}{dx} = -4 + 12x - 3x^2 = 0$

$3x^2 - 12x + 4 = 0$

$x = \frac{12 \pm \sqrt{144-48}}{6} = \frac{12 \pm \sqrt{96}}{6} = \frac{12 \pm 4\sqrt{6}}{6} = 2 \pm \frac{2\sqrt{6}}{3} = 0.367$ or 3.633 (3.d.p.)

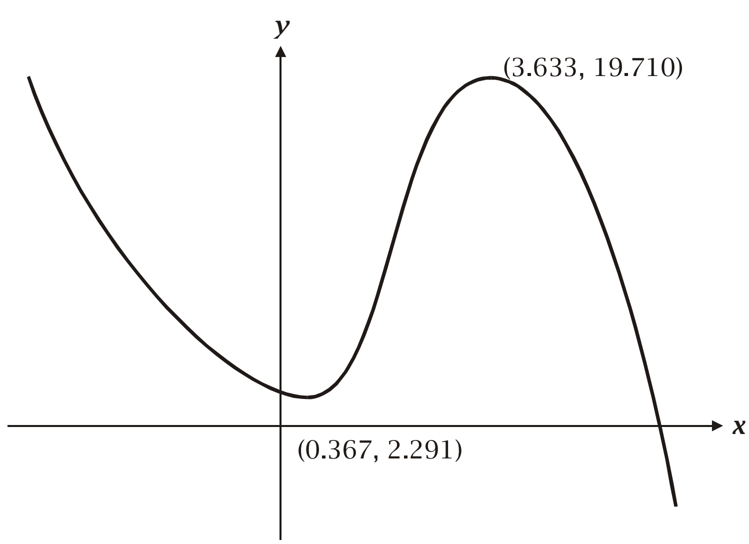
When $x = 0.367$, $y = 2.291$ (3.d.p.)

When $x = 3.633$, $y = 19.710$ (3.d.p.)

$\frac{d^2y}{dx^2} = 12 - 6x$

$\left.\frac{d^2y}{dx^2}\right|_{x=2+\frac{2\sqrt{6}}{3}} < 0$ So $x = 2 + \frac{2\sqrt{6}}{3}$ is a maximum point

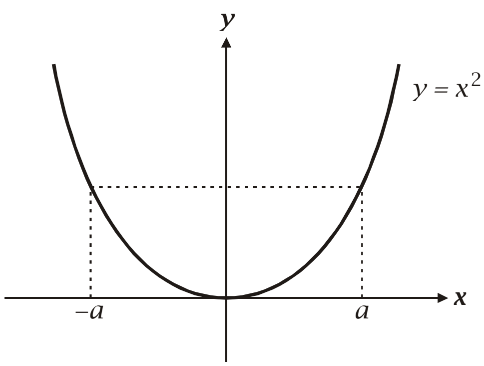
$\left.\frac{d^2y}{dx^2}\right|_{x=2-\frac{2\sqrt{6}}{3}} > 0$ So $x = 2 - \frac{2\sqrt{6}}{3}$ is a minimum point



Inverse of a Function: monotone increasing or decreasing functions

If a function f maps x to y, then the inverse of that function, written f^{-1}, maps y to x. The inverse of a function reverses the process represented by that function. However, not all functions have inverses. This is because functions can be "many-one" or "one-one". An example of a many-one function is

$$f\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\to x^2\end{cases}$$



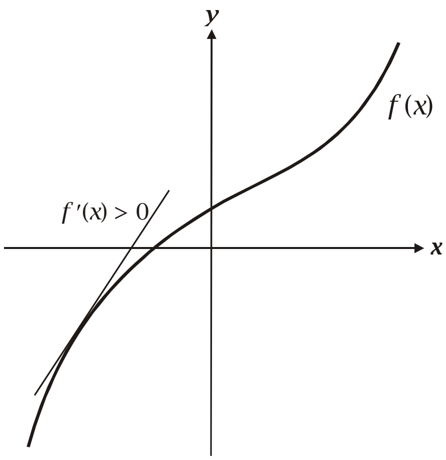
This is many-one because there are two values in the domain giving the same value in the co-domain: $f(a) = f(-a)$. A many-one function is a function such that there are two or more arguments in the domain giving the same value in the co-domain. A many-one function cannot have an inverse because the arguments of the "inverse" would have more than one value, and a function must specify just one value for each argument. A one-one function specifies for each argument just one value. For a function to have an inverse it must be one-one. A one-one function is either always increasing or always decreasing.

An always-increasing function is also called a *monotone increasing* function, and an always-decreasing function is also called a *monotone decreasing* function. To prove that a function is monotone increasing or decreasing we use analytic methods; that is, it is an application of the differential calculus. For a function without points of inflexion to be monotone increasing then its derivative is always positive.

$f(x)$ is monotone increasing $\Leftrightarrow$ for all x: either $f'(x) > 0$
or if $f'(x) = 0$ then $f''(x) = 0$

$f(x)$ is monotone decreasing $\Leftrightarrow$ for all x: either $f'(x) < 0$
or if $f'(x) = 0$ then $f''(x) = 0$

As the definition indicates we will count a function as monotone increasing if it is increasing throughout the domain except, perhaps, where it has a point of inflexion.



Example (5)

Prove that $f(x) = x^3$ is a monotone increasing function.

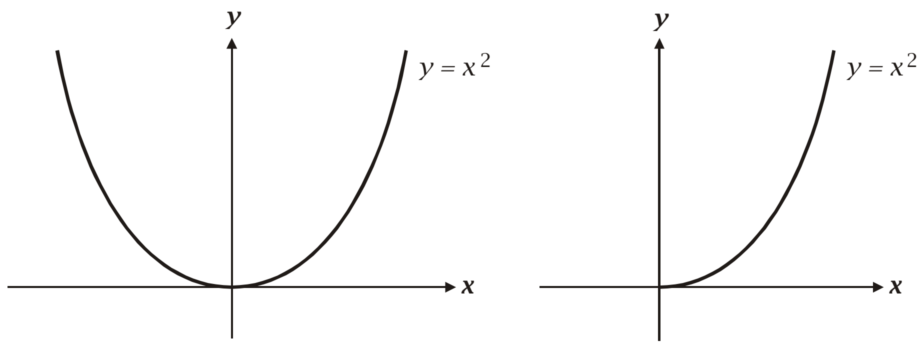
Solution

$f'(x) = 3x^2 > 0$ for all x except $x = 0$

At $x = 0$, $f'(0) = 0$ and $f''(0) = 0$ so this is a point of inflexion.

Hence $f(x)$ is monotone increasing throughout the domain $\mathbb{R}$.

A one-one function can be created from a many-one function by restricting the domain. This means diminishing the size of the domain so that the function becomes increasing or decreasing on the reduced domain. For example



$$f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\to x^2\end{cases}\qquad \xrightarrow[\text{RESTRICTTION}]{} \qquad g:\begin{cases}\mathbb{R}^+\to\mathbb{R}^+\\ x\to x^2\end{cases}$$

The domain is restricted to contain only positive real numbers. The inverse of this restricted function $g(x)=x^2$ is called the square root.

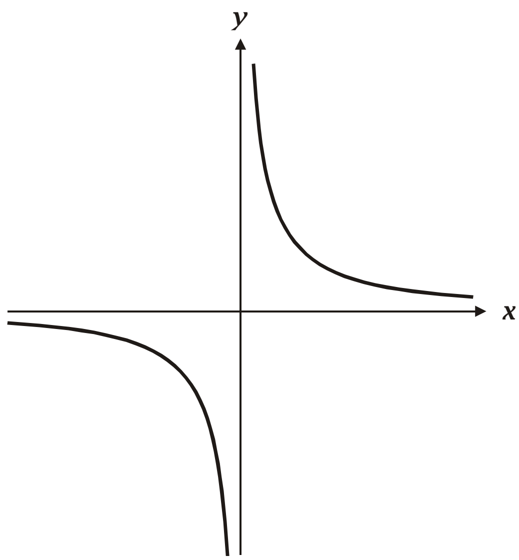
$$g^{-1}:\begin{cases}\mathbb{R}^+\to\mathbb{R}^+\\ x\to\sqrt{x}\end{cases}$$

The reciprocal function - singularities

The reciprocal function takes the form

$$f\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto\dfrac{1}{x}\end{cases}\qquad x\neq 0$$

The graph of the reciprocal function is the rectangular hyperbola.



The graph illustrates important points about this function. At $x = 0$ the function is *undefined* because we cannot divide by zero. The expression $\frac{1}{0}$ is meaningless. The graph shows this because around the origin the function tends to $-\infty$ as we approach $x = 0$ from the negative side, and tends to $+\infty$ as we approach $x = 0$ from the positive side. We say that there is a *singularity* of the function f at $x = 0$, which means that f is undefined there. The y-axis ($x = 0$) is an asymptote of f, but in the neighbourhood of the origin, $x = 0$, the value of $y = f(x)$ *jumps* from $-\infty$ to $+\infty$.

Functions defined piecewise on their domain

Regarding the reciprocal function

$$f\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \dfrac{1}{x}\end{cases}\qquad x\neq 0$$

we could define a function that took a value at $x = 0$. For example

$$g:\ \mathbb{R}\to\mathbb{R}$$
$$\begin{cases}x\mapsto \dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

This definition fills in the missing gap when $x = 0$, though it does so in an arbitrary way and we could have specified that g took any other value when $x = 0$. Furthermore, the definition of g does not "bridge the gap" created by the singularity at $x = 0$ for the reciprocal function f. There is no way to join the two halves of f together so that the singularity is removed. When there is a gap like this we say that the function is *discontinuous*. When there is no "gap" like this, then the function is *continuous*. Whether a function is continuous or discontinuous is important in mathematics and we need to develop rules for determining this property.

The function g has also been constructed from *two* definitions. Each definition applies to a different part of the domain of g. So $g(x)={}^{1}\!/\!_{x}$ on all values of the domain $\mathbb{R}$ with the exception of $x = 0$, and $g(x)=0$ if $x = 0$. The function g comes in two *pieces* and is said to have been defined *piecewise on its domain*. Intuitively, a function is continuous if its graph consists of one unbroken curve. If you were drawing the graph with a pencil you would draw the graph as one curve or line *without lifting your pencil*. This is the intuitive notion of a continuous graph. A function can be discontinuous in basically two different ways. Firstly, when a function contains a reciprocal, then it may have a singularity. The singularity means that the graph has asymptotes around the singularity, so it is not possible to join the two halves of the graph together around the singularity. Singularities create discontinuous graphs. Now there is a second way in which continuity can fail. When we define a function piecewise on its domain, we are joining two functions together. In that case, the functions may either join up continuously, or there may be a discontinuity. This is best shown by example.

Example (6)

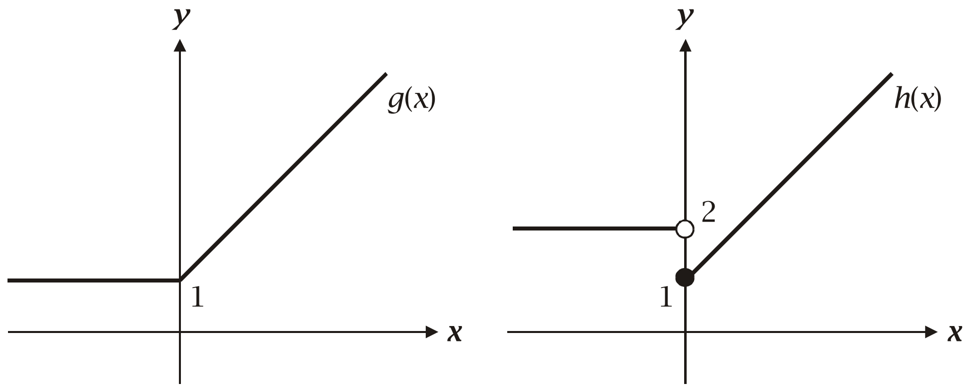
The functions g and h are defined piecewise on the domain $\mathbb{R}$ as follows

$$\begin{cases}g(x)=x+1 & \text{if } x\geq 0\\ g(x)=1 \text{ if } x<0\end{cases}\qquad\qquad \begin{cases}h(x)=x+1 & \text{if } x\geq 0\\ h(x)=2 \text{ if } x<0\end{cases}$$

Sketch the graphs of g and h and explain intuitively why g is continuous and h is not.

Solution

The following sketches show g and h.



The sketch for g shows that g remains as an unbroken "curve" – the two halves are joined together *continuously* at $x = 0$, and g is continuous throughout its domain. The sketch for h employs the convention that an unfilled circle is used to represent a point that *is not included* in the graph, and a filled circle represents a point that *is included* in the graph. When $x = 0$ the function h takes the value 1 because $h(x) = x + 1$ if $x \geq 0$, so the point $(0,1)$ is included in the graph and is shown by the filled circle. When $x < 0$ the function h is defined by $h(x) = 2$ and h takes the value 2 at every point when $x < 0$ *except* $x = 0$. So the point $(0,2)$ is not included in the graph and is represented by the unfilled circle. The graph of h shows intuitively that h is discontinuous at $x = 0$; the graph "jumps" from the value 2 to the value 1 just around $x = 0$. (We say, "in the *neighbourhood* of $x = 0$".) So h is a discontinuous function because the two pieces of h have not been joined together in a way that makes it continuous.

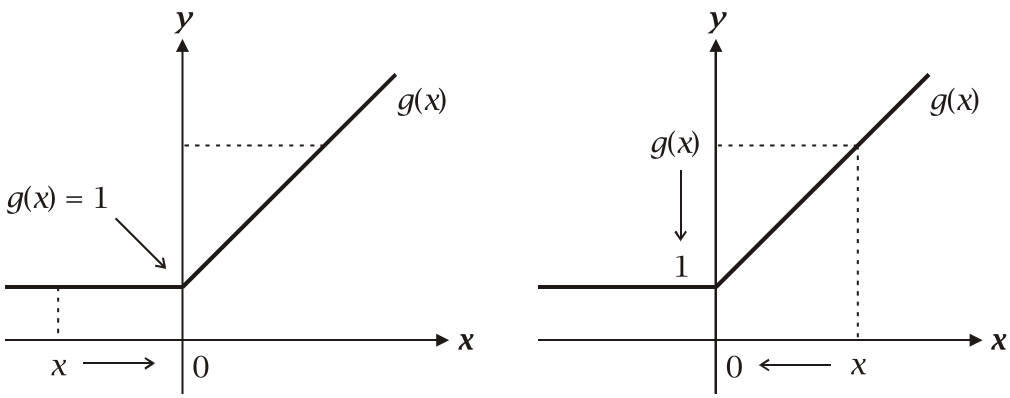
However, in this example we are using "intuitive" arguments to prove that one function is continuous and another is discontinuous in the neighbourhood of some point. We need to replace the intuitive argument by something more analytical. For this purpose we will need to introduce the idea of a *limit*. However, when dealing with arguments about limits we employ two types of argument: *formal* or *rigorous* arguments and *informal* arguments. The formal arguments are quite technical, and in this chapter we will in fact use primarily *informal* arguments about limits to make statements about the continuity of functions. Yet, when we introduce the idea of a limit it is useful to give some idea of what the formal definition of a limit might be. Arguments using formal limits are not dealt with here.

Limits

Let us continue to use the functions g and h defined above.

$$\begin{cases} g(x) = x+1 & \text{if } x \geq 0 \\ g(x) = 1 \text{ if } x < 0 \end{cases} \qquad \begin{cases} h(x) = x+1 & \text{if } x \geq 0 \\ h(x) = 2 \text{ if } x < 0 \end{cases}$$

The intuitive idea behind our arguments about the continuity or discontinuity of these functions involved the notion of a *limit*. As we get *closer and closer* to $x = 0$, the value of $g(x)$ gets closer and closer to 1, whether we approach $x = 0$ from the negative end of the x-axis or from the positive end of it. At $x = 0$ it is unambiguous that $g(x) = 1$.



For this argument we write *informally*

$$\lim_{x\to 0^-} g(x) = 1 \qquad \lim_{x\to 0^+} g(x) = 1 \qquad g(1) = 1$$

Note here the use of – and + symbols in connection with the symbol for the limit. These indicate from which side of the given x-value we are approaching; the minus (–) indicates we approach it from the negative end of the x-axis and the plus (+) sign indicates that we approach it from the positive end of the x-axis. Because the two limits converge on the same value, the function $g(x)$ is continuous.

Before we move on, it is useful to briefly consider what the *formal* definition of a limit might be, and how the *formal* argument would go. Limits are primarily defined for sequences of numbers. For example, you may be familiar with the sequence defined by the general term

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

The sequence is generated by substituting successive values of n

$$u_1 = \left(1+\frac{1}{1}\right)^1 = 2$$

$$u_2 = \left(1+\frac{1}{2}\right)^2 = 1.5^2 = 2.25$$

$$u_3 = \left(1+\frac{1}{3}\right)^3 = 1.333...^3 = 2.370...$$

This sequence can be shown to converge on a single value, and is the definition of the important irrational number *e*.

$$e = \lim_{n\to\infty} u_n = \left(1+\frac{1}{n}\right)^n$$

As *n* gets larger and larger (approaches infinity) the value of $u_n = \left(1+\frac{1}{n}\right)^n$ gets closer and closer to the number $e = 2.71828...$. We can write this also as

$$\left(1+\frac{1}{n}\right)^n \to e \qquad \text{as} \quad n \to \infty .$$

This notation still does not capture formally the notion of getting "closer and closer". This notion involves the idea that the difference between the value of the sequence and the limit becomes smaller and smaller as the sequence progresses. Let us use l to stand for the limit, and the Greek letter ε (epsilon) to stand for a small number. Let u_n stand for the *n*th term of the sequence. Then we are saying that as *n* gets larger and larger

$|u_n - l| < \varepsilon$ whatever ε

That is, however, small we make ε the value of $|u_n - l|$ is smaller than it, provided that *n* is large enough. So this is the basis of the formal definition of a limit

Formal definition of a limit

A function $f(x)$ tends to the limit l as x becomes larger and larger ('tends to infinity') if, when ε is a given positive number, however small, a number *N* can be found, depending on ε such that

$|f(x) - l| < \varepsilon$ whenever $x > N$

This is abbreviated to

$$\lim_{x\to\infty} f(x) = l$$

In this context we are discussing continuity, and the idea that a function defined piecewise may not converge uniquely on a single value, so that if we were drawing the graph of that function we would have to lift the pencil at that point. To capture this idea we modify the above definition as follows.

Formal definition of the limit of a function at a point $x = a$

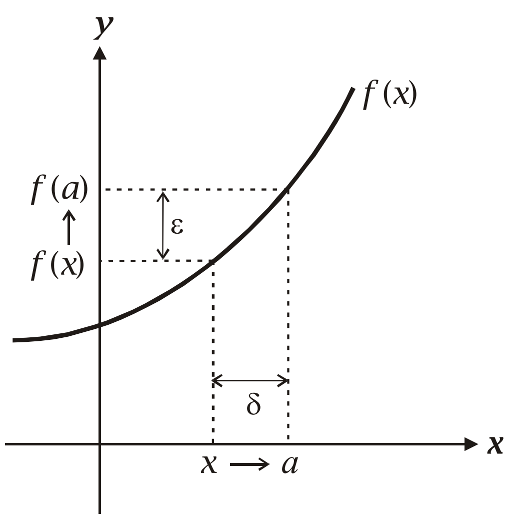
A function $f(x)$ tends to the limit l as x gets closer and closer to a if, when ε is a given positive number, however small, a number δ can be found, depending on ε such that

$|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$

This is abbreviated to

$\lim_{x \to a} f(x) = l$

The following graph illustrates this definition.



Informal arguments

In this chapter "informal" arguments will suffice. That is, we do not use the formal definition of a limit given above, for the reason that the work involved tends to be nasty and often quite unnecessary. Informal arguments are based on the idea that what is obvious is true. For example, it is obvious when $g(x)$ is defined by

$$\begin{cases} g(x) = x + 1 & \text{if } x \geq 0 \\ g(x) = 1 \text{ if } x < 0 \end{cases}$$

that $g(x)$ converges on the value 1 from both directions; that is

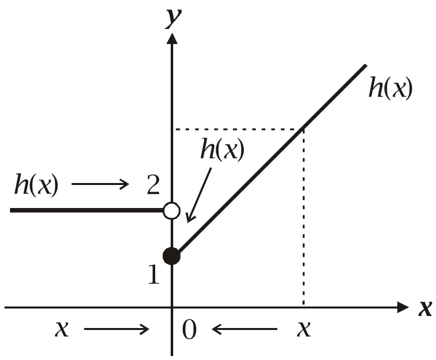
$\lim_{x \to 0^-} g(x) = 1$ $\qquad$ $\lim_{x \to 0^+} g(x) = 1$

However, when $h(x)$ is the function

$$\begin{cases} h(x) = x+1 & \text{if } x \geq 0 \\ h(x) = 2 \text{ if } x < 0 \end{cases}$$

we have

$\lim_{x \to 0^-} h(x) = 2$ $\lim_{x \to 0^+} h(x) = 1$



Since $\lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x)$ this proves *informally* that h is **not** continuous at $x = 0$.

Continuity

We will now explicitly define *continuity*. In this definition the phrase "from above" means that we replace the condition $|x-a| < \delta$ in the definition of a limit by $0 < x - a < \delta$. That is

A function $f(x)$ tends to the limit l *from above* as x gets closer and closer to a if, when ε is a given positive number, however small, a number δ can be found, depending on ε such that

$|f(x) - l| < \varepsilon$ whenever $0 < x - a < \delta$

This is abbreviated to $\lim_{x \to a+} f(x) = l$

The phrase "from below" means that we replace $|x-a| < \delta$ in the definition of a limit by $0 < a - x < \delta$. This is abbreviated to $\lim_{x \to a-} f(x) = l$

Continuity

The function $f(x)$ is *continuous* when $x = a$ if $f(x)$ tends to a limit l as x tends to a from above and to the same limit l as x tends to a from below, while $f(x) = l$ when $x = a$.

In conclusion, to show that a function is continuous at a point a we have to show

(1) $\lim_{x \to a+} f(x) = l$ (2) $\lim_{x \to a-} f(x) = l$ (3) $f(a) = l$

Combining limits

We can combine limits in much the same way that we combine numbers.

Scalar multiple of a limit

Multiplying a function by a constant has the effect of multiplying the corresponding limit by the same constant. In symbols, let a be a real number, then

If $f(n) \to l$ as $n \to \infty$ then $af(n) \to al$ as $n \to \infty$

Alternative notation

$\lim_{x \to a} \{bf(x)\} = b \lim_{x \to a} f(x)$

Sum of limits

The limit of a sum (or difference) of two limits is the sum (or difference) of the limits.

If $f_1(n) \to l_1$ as $n \to \infty$ and $f_2(n) \to l_2$ as $n \to \infty$
then $f_1(n) + f_2(n) \to l_1 + l_2$ as $n \to \infty$

Alternative notation

$\lim_{x \to a} \{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Product of limits

The limit of a product is the product of the limits.

If $f_1(n) \to l_1$ as $n \to \infty$ and $f_2(n) \to l_2$ as $n \to \infty$
then $f_1(n) \times f_2(n) \to l_1 \times l_2$ as $n \to \infty$

Alternative notation

$\lim_{x \to a} \{f(x) \times g(x)\} = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$

Quotient of limits

The limit of a quotient is the quotient of the limits.

If $f_1(n) \to l_1$ as $n \to \infty$ and $f_2(n) \to l_2$ as $n \to \infty$

then $\frac{f_1(n)}{f_2(n)} \to \frac{l_1}{l_2}$ as $n \to \infty$ provided that $l_2 \neq 0$

Alternative notation

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ provided that } \lim_{x\to a} g(x) \neq 0$$

Example (7)

Evaluate $\lim_{x\to 0}\left\{\frac{x^2+3x+2}{x^2-3x+2}\right\}$

Solution

$$\begin{aligned}
\lim_{x\to 0}\left\{\frac{x^2+3x+2}{x^2-3x+2}\right\} &= \frac{\lim_{x\to 0}\left(x^2+3x+2\right)}{\lim_{x\to 0}\left(x^2-3x+2\right)} && \text{[Quotient rule]}\\
&= \frac{\lim_{x\to 0} x^2 + \lim_{x\to 0} 3x + \lim_{x\to 0} 2}{\lim_{x\to 0} x^2 - \lim_{x\to 0} 3x + \lim_{x\to 0} 2} && \text{[Sum rule]}\\
&= \frac{\left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} x\right) + 3\lim_{x\to 0} x + \lim_{x\to 0} 2}{\left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} x\right) - 3\lim_{x\to 0} x + \lim_{x\to 0} 2} && \text{[Product and multiples rules]}\\
&= \frac{0+0+2}{0-0+2} && \text{[Informally evaluating the limits]}\\
&= 1
\end{aligned}$$

Quotients

When one function is divided by another, then we have the possibility of singularities. A singularity may occur whenever a value of x would lead to a zero in the denominator of the quotient. Let

$$h(x) = \frac{f(x)}{g(x)}$$

Suppose that when $x = a$ the function $g(x) = 0$ and suppose also that at this point $f(x) \neq 0$. Then at this point the quotient cannot be defined and $h(x)$ cannot take a value. Strictly speaking the point $x = a$ cannot belong to the domain of $h(x)$ and we should specify that $h(x)$ is defined on a domain that *does not include* $x = a$. Alternatively, we could add a value for $h(a)$ by defining it piecewise; for example, we could arbitrary set $h(a)=1$ and define the function by the following.

$$h(x)=\frac{f(x)}{g(x)} \qquad \text{whenever } x \neq a$$
$$h(x)=1 \qquad \text{if } x = a$$

In practice we usually speak of a function $h(x)$ being defined on the whole domain (say $\mathbb{R}$) and assume that it is clear from context that $h(x)=\frac{f(x)}{g(x)}$ is undefined at any value where $g(x)=0$.

Example (8)

Name any points at which the following functions are discontinuous

(a) $\frac{x}{x^2-3x+2}$ (b) $\frac{1}{\sin x}$

Solution

(a) The function $\frac{x}{x^2-3x+2}$ will be discontinuous (and undefined) whenever the denominator

$x^2-3x+2=(x-1)(x-2)$

is zero. That is, at $x = 1$ and $x = 2$.

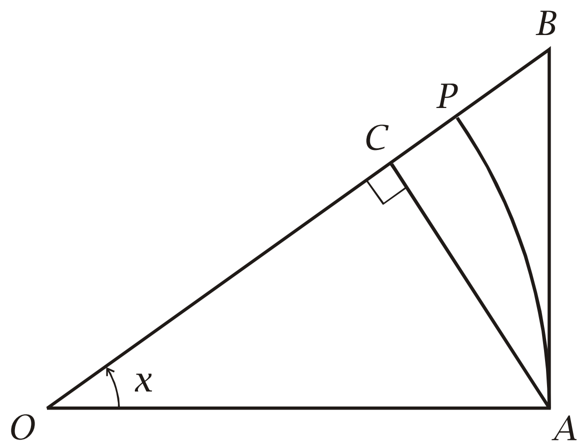
(b) The function $\frac{1}{\sin x}$ will be discontinuous (and undefined) whenever the denominator $\sin x$ is zero. That is at $n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$

As (a) above indicates there exists the possibility that a quotient may have a limit even at a point where the denominator takes a zero value; that is, if the numerator simultaneously is zero at this point. However, this alone would not be enough to show that a quotient was continuous, because the function has to converge from above and below on zero, and not just equal zero at that point. Nonetheless, here is an important example of when a quotient has a limit and is continuous at a given point, even though the denominator is zero at that point.

Example (9)

Prove that the function $f(x)=\frac{\sin x}{x}$ is continuous at $x = 0$. Furthermore, show that the limit as $x \to 0$ of $\frac{\sin x}{x}$ is unity. That is $\lim_{x\to 0}\left\{\frac{\sin x}{x}\right\}=1$

Proof



In the right-angled triangle OAB let OA be of unit length, and the angle $\angle BOA$ be x radians, where x is small. Let AC be perpendicular to OB, and AP be an arc of a circle. Then by geometric intuition

$AC < \text{arc } AP < AB$

But $AC = \sin x$, arc $AP = x$, $AB = OB\sin x$

Hence

$\sin x < x < OB\sin x$

$$1 < \frac{x}{\sin x} < OB$$

As $x \to 0$, OB tends to equality with OA, hence

$$\lim_{x\to 0} OB = 1$$

We have

$$1 < \lim_{x\to 0}\frac{x}{\sin x} < \lim_{x\to 0} OB$$

That is

$$1 < \lim_{x\to 0}\frac{x}{\sin x} < 1$$

Hence

$$\lim_{x\to 0}\frac{x}{\sin x} = 1$$

This result is required when it is proven that the derivative of sin x is cos x.

Image set

We may wish to know what the image of a subset A of the domain of a function is. The *image of a set* A *under the function f* is the set of all values to which elements in A are mapped by *f*. The symbol $f(A)$ is used to denote this set and is defined as

$f(A) = \{f(x) : x \in A \text{ provided that } f(x) \text{ is defined}\}$

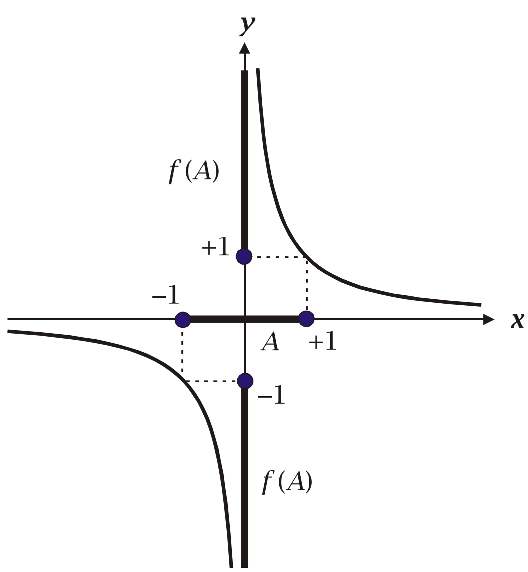
We add the proviso that $f(x)$ must be defined for all values $x \in A$. For example, the function $f(x) = \frac{1}{x}$ does not take a value when $x = 0$, because that is equivalent to dividing by zero.

Example (10)

Find the image under *f* of the set A given by interval $[-1,1]$ when $f(x) = \frac{1}{x}$.

Solution

The graph of $f(x) = \frac{1}{x}$ on the set A is



As the graph illustrates

$\lim_{x \to a+} f(x) = +\infty$

$\lim_{x \to a-} f(x) = -\infty$

$f(x)$ is undefined when $x = 0$

On the interval $[-1,0)$ $f(x)$ is a monotone decreasing function, and on the interval $(0,1]$ $f(x)$ is also monotone decreasing. We can show this by showing that the first derivative $f'(x) = -x^{-2}, x \neq 0$ never takes zero, so the graph of $f(x)$ has no turning points; furthermore, $f'(x) < 0$ for all values of x $(x \neq 0)$ so $f(x)$ is always decreasing. Since

$f(1) = 1$
$f(-1) = -1$

the image of A under f is the set

$f(A) = [-1, -\infty) \cup [1, \infty)$

Inverse image

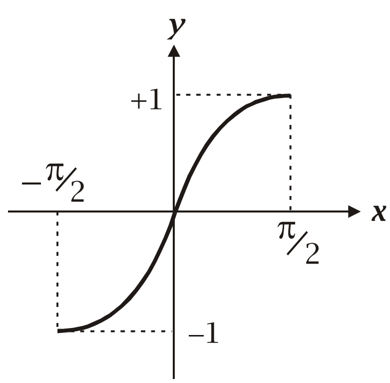
When a function f has an inverse f^{-1} then the image of a set A under the inverse may also be found. This is the set

$f^{-1}(B) = \{x : f(x) \text{ is defined and } f(x) \in B\}$

Example (11)

Find the inverse image of the set $B = [0,1]$ of the function $f(x) = \sin x$

Solution



The inverse of f is the inverse of the restricted function

$$f_1 : \begin{cases} \left[-\pi/2, \pi/2\right] \to [-1,1] \\ x \to \sin x \end{cases}$$

Its inverse is

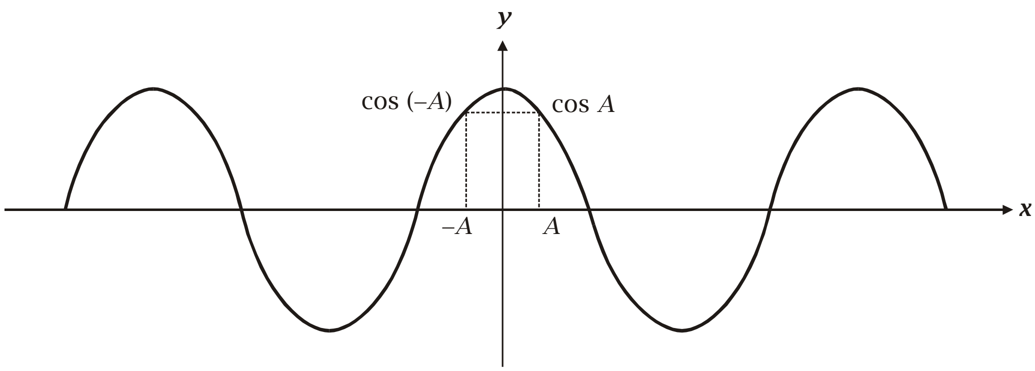
$$g^{-1}\begin{cases}[-1,1]\to\left[-\pi/2,\pi/2\right]\\ \quad x\to\sin^{-1}x\end{cases}$$

On the set $B=[0,1]$ the image set is

$$g^{-1}(B)=\{x:g(x)\text{ is defined and }g(x)\in B\}$$
$$=\left[0,\pi/2\right]$$

Odd and even functions

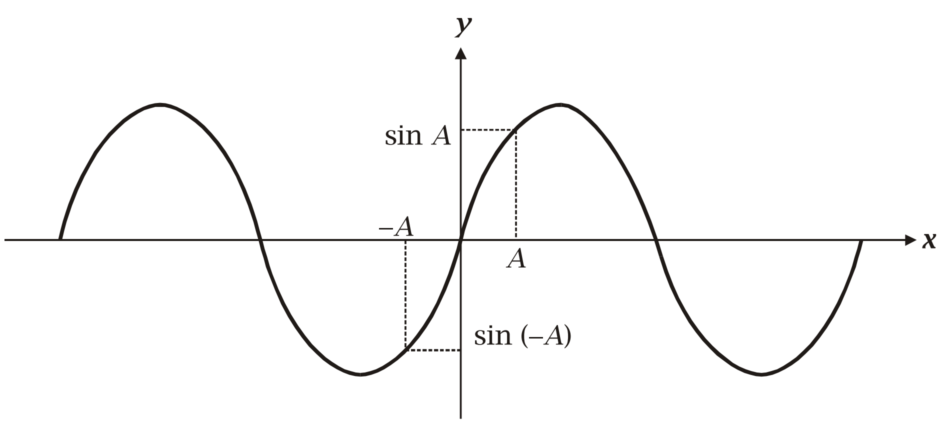
The graphs of some functions exhibit symmetry. For example $y=\cos x$ is symmetrical about the y-axis.



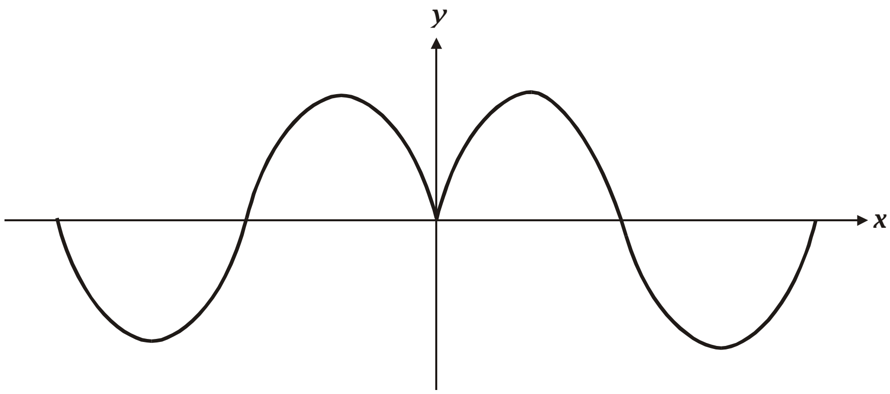
Such functions are said to be *even* or *symmetric* functions. The formal criterion for an even function is

$f(x)$ is even $\Leftrightarrow f(x)=f(-x)$

A function that is not even may yet be "almost" symmetrical. For example, in the graph of $y=\sin x$



If we were to reflect the negative part in the *x*-axis we would obtain an even function.



Example (12)

Prove that the function

$$\begin{cases} f(x) = \sin x & \text{if} \quad x \geq 0 \\ f(x) = -\sin x & \text{if} \quad x < 0 \end{cases}$$

is an even function.

Solution

Suppose $x \geq 0$ then $-x < 0$ and $f(-x) = -\sin(-x) = \sin x = f(x)$

Suppose $x < 0$ then $-x > 0$ and $f(-x) = \sin(-x) = -\sin x = f(x)$

In either case

$f(-x) = f(x)$

So *f* is an even function.

Functions like $f(x) = \sin x$ that can be turned into even functions in this way are called *odd* or *anti-symmetric.* The formal criterion for an odd function is

$f(x)$ is odd $\Leftrightarrow f(-x) = -f(x)$

Example (13)

Prove that the function

$f(x) = x(x+1)(x-1)$

is odd.

Solution

$$\begin{aligned} f(-x) &= (-x)((-x)+1)((-x)-1) \\ &= (-x)(1-x)(-x-1) \\ &= -x(x-1)(x+1) \\ &= -f(x) \end{aligned}$$

So f is odd.

Functions that do not match either criterion are called *neither odd not even.*